Warm-Up!

1. Since $BA:AC = 3:2$, it follows that $AC = \frac{2}{3}BC$. Therefore, $AC = \frac{2}{3} \times 45 = 30$.

2. Since $AD = 5$ units and $AB = BC = 2AD$, it follows that $AB = BC = 2(5) = 10$ units. If we draw segment $DE$, parallel to side $AB$ and intersecting segment $BC$ at point $E$, as shown, rectangle $ABED$ is formed with side lengths $AD = BE = 5$ units and $AB = DE = 10$ units, and with area is $10 \times 5 = 50$ units$^2$. So, right triangle $DEC$, with leg lengths $DE = 10$ units and $CE = 10 - 5 = 5$ units, has area $(1/2) \times 10 \times 5 = 25$ units$^2$. The total area of trapezoid $ABCD$, then, is $50 + 25 = 75$ units$^2$.

3. Since segments $MN$ and $OP$ are parallel, we can conclude that $\triangle MNQ \sim \triangle POQ$ (Angle-Angle). Therefore, the ratios of corresponding sides of the triangles are congruent. Since $ON = 24$ units, it follows that $OQ = 24 - NQ$. We can set up the following proportion: $NQ/(24 - NQ) = 12/20$. Cross-multiplying and solving for $NQ$, we get $20(NQ) = 12(24 - NQ) \rightarrow 20(NQ) = 288 - 12(NQ) \rightarrow 32(NQ) = 288 \rightarrow NQ = 9$ units.

4. A segment drawn connecting $A$ with $E$, as shown, creates a right triangle $ADE$ and trapezoid $ABCE$. Triangle $ADE$ has legs of length 10 units and 2 units, making its area $(1/2) \times 10 \times 2 = 10$ units$^2$. The area of trapezoid $ABCE$ is difference between the area rectangle $ABCD$ and the area of triangle $ADE$. Rectangle $ABCD$ has area $10 \times 6 = 60$ units$^2$. So, the area of trapezoid $ABCE$ is $60 - 10 = 50$ units$^2$. The ratio of the area of triangle $ADE$ to the area of trapezoid $ABCE$, then, is $10/50 = 1/5$.

The Problems are solved in the MATHCOUNTS$^\text{M}in\text{s}$ video.

Follow-up Problems

5. We are asked to determine the area of the shaded quadrilateral, which happens to be a trapezoid. The height of the trapezoid is 4, the side length of the middle square. Notice that triangles $MNO$, $MPR$ and $MST$, shown in the figure, are similar right triangles. The base $NO$ of the trapezoid is the shorter leg of $\triangle MNO$ and the base $PR$ is the shorter leg of $\triangle MPR$. Triangle $MST$ has a shorter leg of length $ST = 6$ and a longer leg of length $MS = 2 + 4 + 6 = 12$.

The ratio of the lengths of the shorter leg to the longer leg is $1:2$. The longer leg of $\triangle MPR$ has length $MP = 2 + 4 = 6$, so its shorter leg must have length $PR = 1/2 \times 6 = 3$. The longer leg of $\triangle MNO$ has length $MN = 2$, so its shorter leg must have length $NO = 1/2 \times 2 = 1$. Therefore, the trapezoid has area $1/2 \times (1 + 3) \times 4 = 8$ units$^2$. 

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6. Since $2(0) - 4 = -4$, it follows that $y = 2x - 4$ intersects the $y$-axis at $A(0, -4)$. Similarly, since $-3(0) + 16 = 16$, it follows that $y = -3x + 16$ intersects the $y$-axis at $B(0, 16)$. These two lines intersect each other when $2x - 4 = -3x + 16 \rightarrow x = 4$ and $y = 2(4) - 4 = 8 - 4 = 4$, which we'll label $C(4, 4)$. As the figure shows, the interior region formed by $y = 2x - 4$, $y = -3x + 16$ and the $y$-axis is $\triangle ABC$. The dashed segments show the extension of $y = -3x + 16$ beyond point $C$ and a portion of the horizontal line $y = -4$. The intersection of these dashed segments occurs when $-4 = -3x + 16 \rightarrow 3x = 20 \rightarrow x = \frac{20}{3}$, at a point we'll label $D\left(\frac{20}{3}, -4\right)$. As the figure shows, the area of $\triangle ABD$ minus $\triangle ACD$. Triangles $ABD$ and $ACD$ both have base length $AD = \left|\frac{20}{3} - 0\right| = \frac{20}{3}$ units. Right triangle $ABD$ has height $|16 - (-4)| = 20$ units, and $\triangle ACD$ has height $|4 - (-4)| = 8$ units. So, $\triangle ABC$ has area $(1/2)(\frac{20}{3})(20) - (1/2)(\frac{20}{3})(8) = \left(\frac{10}{3}\right)(20 - 8) = \left(\frac{10}{3}\right)(12) = (10)(4) = 40$ units$^2$.

7. Triangle $BCD$ is a 30-60-90 right triangle with a shorter leg of length 6. Based on properties of 30-60-90 right triangles, segment $BC$, the longer leg, has length $6\sqrt{3}$. Since $M$ is the midpoint of segment $AD$, $MD = 6\sqrt{3}/2 = 3\sqrt{3}$. For right triangle $CDM$, we know $CD = 6$ and $DM = 3\sqrt{3}$, so we can use the Pythagorean Theorem to determine $CM$. We have $CM^2 = 6^2 + (3\sqrt{3})^2 \rightarrow CM = \sqrt{36 + 27} \rightarrow CM = \sqrt{(63)} \rightarrow CM = 3\sqrt{7}$. If $m\angle DBC = 30^\circ$, then $m\angle BDA = 30^\circ$ because they are alternate interior angles. Also $m\angle CKB = m\angle MKD$ since they are vertical angles. That means $\triangle CKB \sim \triangle MKD$, and $BC/DM = CK/MK$. Substituting and simplifying $BC/DM$, we have $2/1 = CK/MK$, which means $MK = \frac{1}{2}CM \rightarrow MK = \frac{1}{2} \times 3\sqrt{7} \rightarrow MK = \sqrt{7}$.

8. From the figure, we can see that the area of $\triangle ACD$ is the sum of the areas of $\triangle ABE$ and trapezoid $BCDE$. Also, we are told that the area of trapezoid $BCDE$ is 8 times the area of $\triangle ABE$. It follows that the area of $\triangle ACD$ is 9 times the area of $\triangle ABE$. That means the ratio of sides $BE$ and $CD$ is $\sqrt{1/9} = 1/3$. Since segments $BE$ and $CD$ are also sides of triangles $EBX$ and $CDX$, respectively, it follows that the ratio of the areas of $\triangle EBX$ and $\triangle CDX$ is $1^2/3^2 = 1/9$. The problem states that the area of $\triangle CDX$ is 27 units$^2$, so the area of $\triangle EBX$ is $(1/9) \times 27 = 3$ units$^2$. Using the method from the video, we can determine the areas of $\triangle BCX$ and $\triangle DEX$ by multiplying $\sqrt{3} \times \sqrt{27} = \sqrt{81} = 9$. Therefore, $\triangle BCX$ and $\triangle DEX$ each have an area of 9 units$^2$. We now can calculate the area of trapezoid $BCDE$ to be $3 + 27 + 9 + 9 = 48$ units$^2$. Using Harvey’s trick results in the same answer since $(\sqrt{3} + \sqrt{27})^2 = (\sqrt{3} + 3\sqrt{3})^2 = (4\sqrt{3})^2 = 48$ units$^2$. So the area of $\triangle ABE$ is $(1/8) \times 48 = 6$ units$^2$. Thus, the area of $\triangle ACD$ is $48 + 6 = 54$ units$^2$. This also confirms our assertion that the area of $\triangle ACD$ is 9 times the area of $\triangle ABE$ since $9 \times 6 = 54$ units$^2$. 

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