Warm-Up!

1. There are many rectangles that can be drawn in triangle ABC with one vertex at A and each of the other three vertices on a side of the triangle. While each of these rectangles has exactly the same perimeter, they do not all have the same area. We are asked to determine the largest possible area of one the of these rectangles. Consider, for a moment, the five rectangles with integer dimensions 1×5, 2×4, 3×3, 4×2, 5×1. The perimeter of each of these rectangles is 12 units, but they have areas 5, 8, 9, 8 and 5 units^2, respectively. Notice that the closer in measure the length and width are, the greater the area of the rectangle, and that the rectangle in which the length and width are equal has the greatest area. So, the rectangle with the largest area has one vertex at A and the other three vertices at the midpoints of each side. It is a square with side length 3 units and area **9** units^2.

The Problems are solved in the video.

Follow-up Problems

2. As demonstrated in the video, rectangles can be constructed in the triangle ranging in size from very tall (height nearly equal to the height of the triangle measured from vertex A to base BC) and narrow (width nearly equal to zero) to short (height nearly equal to zero) and very wide (width nearly equal to the length of base BC). The rectangle with the largest possible area has height equal to half the height of the triangle measured from vertex A to base BC, and width equal to half the length of base BC. The height of the triangle measured from vertex A to base BC is $3\sqrt{2}$ units. So, the rectangle with the greatest area has height $3\sqrt{2}/2$ units and width $6\sqrt{2}/2 = 3\sqrt{2}$ units. Therefore, the largest possible area of one of these rectangles is $3\sqrt{2}/2 \times 3\sqrt{2} = (3\sqrt{2})^2/2 = 18/2 =$ **9** units^2, which is the same area we calculated in problem #1. This is no coincidence since the triangles in the two problems are identical, and it was proven in the video that the largest possible area of a rectangle inscribed in a triangle is half the area of the triangle.

3. Consider triangle BCX inscribed in rectangle ABCD, as shown in Figure 1. Side BC, which is 12 units, also is the base of the triangle. In this case, since the third vertex of the triangle is on side AD, the height of the triangle is the same as the length of side AB, 6 units.

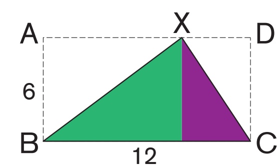


Figure 1

Now consider triangle ABY inscribed in rectangle ABCD, as shown in Figure 2. Here, side AB, which is 6 units, also is the base of the triangle. Since the third vertex of the triangle is on side CD, the height of this triangle is the same as the length of side BC, 12 units. In both of these cases, because each triangle's base and height are the same dimensions as the rectangle, they both have area equal to half the area of the rectangle or $1/2(6)(12) = 36$ units^2. In fact, any triangle inscribed in rectangle ABCD with a base that is a side of the rectangle and the third vertex on the opposite side of the rectangle will have area equal to half that of the rectangle.

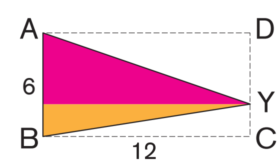


Figure 2

Finally, consider triangle BXY inscribed in rectangle ABCD, as shown in Figure 3. No side of this triangle coincides with a side of rectangle ABCD. We need to determine if this triangle has an area that is greater than half the area of rectangle ABCD. Using the technique from the video, we can show that the combined area of triangles ABX, BCY and DXY is greater than the area of triangle BXY. Therefore, the area of triangle BXY must be less than half the area of the rectangle. In fact, any triangle inscribed in rectangle ABCD such that no side coincides with a side of the rectangle will have an area less than half that of rectangle ABCD. Therefore, the largest possible area of a triangle inscribed in rectangle ABCD is **36** units^2.

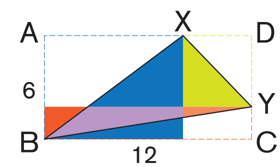
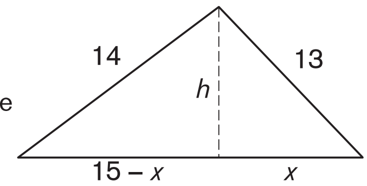
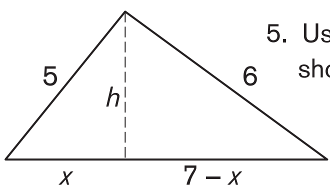


Figure 3

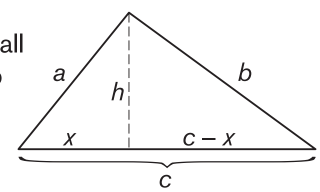
© 2014 MATHCOUNTS Foundation. All rights reserved. MATHCOUNTS Mini Solution Set



4. Using the Pythagorean Theorem, we can write the following equations for the triangle shown: $h^2 + (15 - x)^2 = 196$ and $h^2 + x^2 = 169$. Subtracting these two equations, we get $h^2 + (15 - x)^2 - h^2 - x^2 = 196 - 169 \rightarrow (15 - x)^2 - x^2 = 27$. Simplying and solving for x gives us $225 - 30x + x^2 - x^2 = 27 \rightarrow 225 - 30x = 27 \rightarrow 30x = 198 \rightarrow x = 198/30 = 33/5$ units. We can substitute for x in the equation $h^2 + x^2 = 169$ to get $h^2 + (33/5)^2 = 169$. When we simplify and solve for h, we get $h^2 + 1089/25 = 169 \rightarrow h^2 = 169 - 1089/25 \rightarrow h^2 = 3136/25 \rightarrow h = 56/5$ units. So this triangle, with base length 15 units and height 56/5 units, has area $(1/2)(15)(56/5) = \mathbf{84}$ units^2.



5. Using the Pythagorean Theorem, we can write the following equations for the triangle shown: $h^2 + (7 - x)^2 = 36$ and $h^2 + x^2 = 25$. Subtracting these two equations, we get $h^2 + (7 - x)^2 - h^2 - x^2 = 36 - 25 \rightarrow (7 - x)^2 - x^2 = 11$. Simplying and solving for x gives us $49 - 14x + x^2 - x^2 = 11 \rightarrow 49 - 14x = 11 \rightarrow 14x = 38 \rightarrow x = 38/14 = 19/7$ units. We can substitute for x in the equation $h^2 + x^2 = 25$ to get $h^2 + (19/7)^2 = 25$. When we simplify and solve for h, we get $h^2 + 361/49 = 25 \rightarrow h^2 = 25 - 361/49 \rightarrow h^2 = 864/49 \rightarrow h = (12\sqrt{6})/7$ units. So this triangle, with base length 7 units and height $(12\sqrt{6})/7$ units, has area $(1/2)(7)((12\sqrt{6})/7) = \mathbf{6\sqrt{6}}$ units^2.



6. Let a, b and c represent the side lengths of a triangle, and let h represent its height, all measured in units. As shown in the figure, the altitude splits the base of the triangle into two segments of length x units and $c - x$ units, and divides the original triangle into two right triangles. As demonstrated in the video, we can use the Pythagorean Theorem to write the following equations:

$$h^2 + (c - x)^2 = b^2 \qquad\qquad h^2 + x^2 = a^2 \quad \textbf{[2]}$$
$$h^2 + c^2 - 2cx + x^2 = b^2 \quad \textbf{[1]}$$

When we subtract [2] from [1], we get a third equation, which we then can solve for x:

$$\begin{array}{rl} h^2 + c^2 - 2cx + x^2 = & b^2 \\ -h^2 \qquad\qquad -x^2 = & -a^2 \\ \hline c^2 - 2cx \qquad = & b^2 - a^2 \end{array} \quad \rightarrow \quad \begin{aligned} c^2 - 2cx &= b^2 - a^2 \\ a^2 - b^2 + c^2 &= 2cx \\ \frac{a^2 - b^2 + c^2}{2c} &= x \quad \textbf{[3]} \end{aligned}$$

The area of a triangle equals (1/2) × base × height. We know the base has length c units, so let's find h in terms of a, b and c. To start, we'll rearrange [2] and replace x with the expression in [3].

$$\begin{aligned} h^2 + x^2 &= a^2 \\ h^2 &= a^2 - x^2 \\ h^2 &= (a + x)(a - x) \end{aligned} \quad \rightarrow \quad \begin{aligned} h^2 &= \left[a + \frac{a^2 - b^2 + c^2}{2c}\right]\left[a - \frac{a^2 + b^2 - c^2}{2c}\right] \\ h^2 &= \left[\frac{2ac + a^2 - b^2 + c^2}{2c}\right]\left[\frac{2ac - a^2 + b^2 - c^2}{2c}\right] \end{aligned}$$

Recall that $(a + c)^2 = a^2 + 2ac + c^2$ and $(a - c)^2 = a^2 - 2ac + c^2$, so we can substitute to get

$$h^2 = \left[\frac{a^2 + 2ac + c^2 - b^2}{2c}\right]\left[\frac{b^2 - (a^2 - 2ac + c^2)}{2c}\right] = \left[\frac{(a + c)^2 - b^2}{2c}\right]\left[\frac{b^2 - (a - c)^2}{2c}\right] =$$

$$\frac{(a + c - b)(a + c + b)(b - a + c)(b + a - c)}{4c^2} = \frac{(a + c + b)(-a + b + c)(a - b + c)(a + b - c)}{4c^2} \quad \textbf{[4]}$$

© 2014 MATHCOUNTS Foundation. All rights reserved. MATHCOUNTS Mini Solution Set

The semiperimeter, s, of a triangle is half of the triangle's perimeter. For this particular triangle, $s = (a + b + c)/2$. This equation can be rewritten a number of different ways to find expressions equivalent to those in the numerator of [4].

Multiplying both sides of the equation by 2 gives us $2s = a + b + c$ **[6]**.

Subtracting $2a$ from each side of [6], we get

$$2s - 2a = a + b + c - 2a$$
$$2(s - a) = -a + b + c \qquad \textbf{[7]}$$

Subtracting $2b$ from each side of [6], we get

$$2s - 2b = a + b + c - 2b$$
$$2(s - b) = a - b + c \qquad \textbf{[8]}$$

Subtracting $2c$ from each side of [6], we get

$$2s - 2c = a + b + c - 2c$$
$$2(s - c) = a + b - c \qquad \textbf{[9]}$$

Now we can substitute these values for the corresponding expression in the numerator of [4]. Doing so yields the following equation:

$$h^2 = \frac{(2s)2(s - a)2(s - b)2(s - c)}{4c^2}$$

$$h^2 = \frac{16s(s - a)(s - b)(s - c)}{4c^2}$$

$$h^2 = \frac{4s(s - a)(s - b)(s - c)}{c^2}$$

Taking the square root of each side, we see that

$$h = \sqrt{\frac{4s(s - a)(s - b)(s - c)}{c^2}}$$

$$h = \frac{2\sqrt{s(s - a)(s - b)(s - c)}}{c}$$

$$ch = 2\sqrt{s(s - a)(s - b)(s - c)} \qquad \textbf{[10]}$$

As previously stated, the area of a triangle equals (1/2) × base × height, which is, in this case, (1/2)ch. Replacing ch with the expression in [10], we have shown that

$$\mathbf{A} = \sqrt{s(s - a)(s - b)(s - c)}$$

© 2014 MATHCOUNTS Foundation. All rights reserved. MATHCOUNTS Mini Solution Set