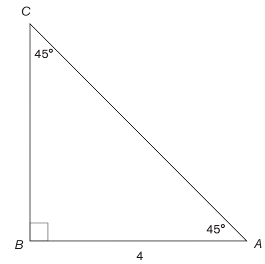
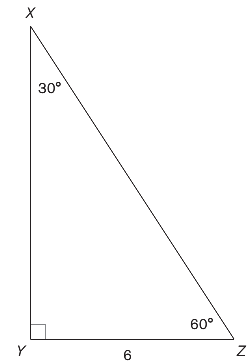
April 2010 Activity Solutions

Warm-Up!



1. Because angles A and C have the same measure, the sides opposite these angles will also be equal. So, since the length of AB is 4, the length of BC is also **4**. By laws of 45-45-90 right triangles, the hypotenuse AC then has length **$4\sqrt{2}$**.

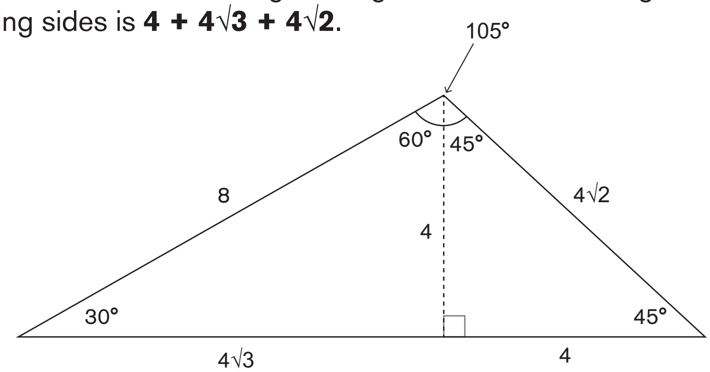


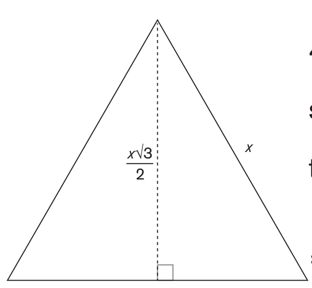
2. By laws of 30-60-90 right triangles, given $YZ = 6$, we know that XZ must equal 2(6) = **12**, and XY = **$6\sqrt{3}$**.

The Problems are solved in the MATHCOUNTS® Minis video.

Follow-up Problems

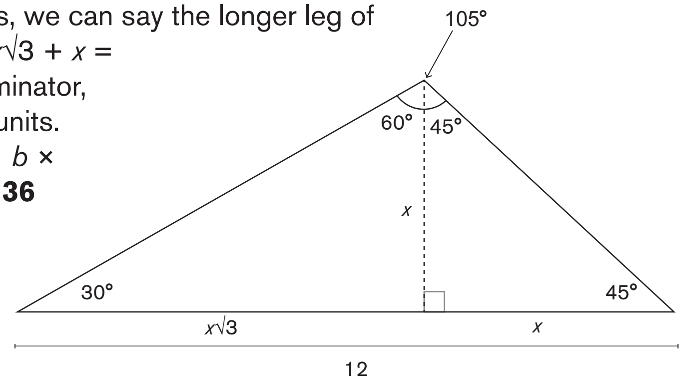
3. As shown in the image, we can determine the lengths of the other two sides by creating two right triangles within the given triangle. By laws of 30-60-90 right triangles, the short leg of this triangle is $1/2 \times 8 = 4$ units in length. The long leg is then $4\sqrt{3}$ units in length. The short leg of the 30-60-90 right triangle is shared with the 45-45-90 right triangle, so the legs of the 45-45-90 right triangle are each 4 units in length. This means that the hypotenuse of the 45-45-90 right triangle is $4\sqrt{2}$ units in length. Thus, the sum of the lengths of the two remaining sides is **$4 + 4\sqrt{3} + 4\sqrt{2}$**.



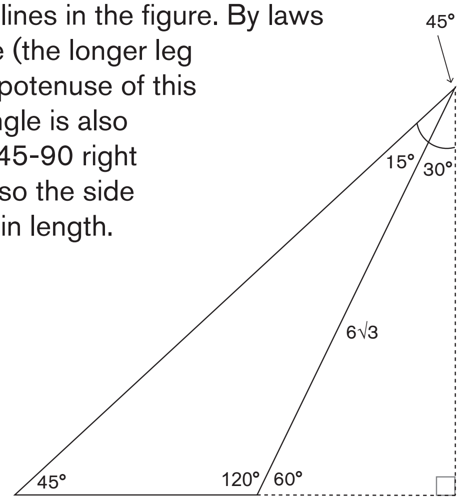


4. Dividing an equilateral triangle in half creates two 30-60-90 right triangles, as shown. By laws of 30-60-90 right triangles, we find the height of an equilateral triangle with side length x to be $\frac{x\sqrt{3}}{2}$. So, the area of this equilateral triangle is $A = (1/2) \times b \times h = (1/2) \times x \times \frac{x\sqrt{3}}{2} = \frac{\sqrt{3}}{4}x^2$.

5. As 12 units is the given length of the base of the triangle, we will only need to find the height, x, of this triangle in order to determine the triangle's area. The dashed line in the figure, the height, creates two right triangles. As the height is of length x, we can say the other leg of the 45-45-90 right triangle is also length x by laws of 45-45-90 right triangles. The height is shared with the 30-60-90 right triangle as well, so by laws of 30-60-90 right triangles, we can say the longer leg of this triangle is of length $x\sqrt{3}$. Now, we see that $12 = x\sqrt{3} + x = x(\sqrt{3} + 1)$. After isolating x and rationalizing the denominator, we find that the height of the triangle, x, is $6(\sqrt{3} - 1)$ units. Now, we can find the area of the triangle: $A = (1/2) \times b \times h = (1/2) \times 12 \times 6(\sqrt{3} - 1) = 36(\sqrt{3} - 1) = \mathbf{36\sqrt{3} - 36}$ units^2.



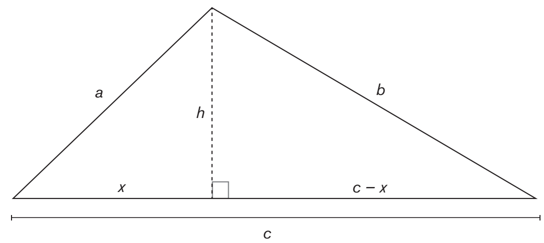
6. We can use the given triangle to create a larger 45-45-90 right triangle, as well as a 30-60-90 right triangle outside of the given triangle, as shown by the dashed lines in the figure. By laws of 30-60-90 right triangles, we can say that the vertical dashed line (the longer leg of the 30-60-90 right triangle) is 9 units in length, given that the hypotenuse of this triangle is $6\sqrt{3}$ units long. The longer leg of the 30-60-90 right triangle is also one of the legs of the 45-45-90 right triangle. Thus, by laws of 45-45-90 right triangles, the hypotenuse of the 45-45-90 right triangle, which is also the side of the original triangle opposite the 120-degree angle, is $\mathbf{9\sqrt{2}}$ units in length.



7. The semiperimeter of an equilateral triangle with side length 6 is $(6 + 6 + 6)/2 = 18/2 = 9$. Using Heron's formula, we find the area of this triangle to be $A = \sqrt{[9(9 - 6)(9 - 6)(9 - 6)]} = \sqrt{(9 \times 3^3)} = \sqrt{(3^5)} = \mathbf{9\sqrt{3}}$ units^2. Using the formula we found in Problem 4 for the same triangle, we find the area to be $A = \sqrt{(3)}/4 \times 6^2 = \sqrt{(3)}/4 \times 36 = \mathbf{9\sqrt{3}}$ units^2. Both area formulas give the same result.

8. We could use the Pythagorean theorem to find the length of the hypotenuse of this right triangle, but you may also know that 7-24-25 is a Pythagorean Triple. So, the three side lengths of this triangle are 7, 24 and 25. Thus, its semiperimeter is $(7 + 24 + 25)/2 = 28$. Using Heron's formula, we find the area of this triangle to be $A = \sqrt{[28(28 - 7)(28 - 24)(28 - 25)]} = \sqrt{(28 \times 21 \times 4 \times 3)} = \sqrt{(4 \times 7 \times 3 \times 7 \times 4 \times 3)} = 4 \times 3 \times 7 = \mathbf{84}$ units^2.

9. Let a, b and c represent the side lengths of a triangle, and let h represent its height, all measured in units. As shown in the figure, the altitude splits the base of the triangle into two segments of length x units and $c - x$ units, and divides the original triangle into two right triangles. As demonstrated in the video, we can use the Pythagorean theorem to write the following equations:



$$h^2 + (c - x)^2 = b^2$$
$$h^2 + c^2 - 2cx + x^2 = b^2 \qquad [1]$$

$$h^2 + x^2 = a^2 \qquad [2]$$

When we subtract [2] from [1], we get a third equation, which we then can solve for x:

$$\begin{array}{r} h^2 + c^2 - 2cx + x^2 = \ \ b^2 \\ -h^2 \qquad\qquad\quad -x^2 = -a^2 \\ \hline c^2 - 2cx \qquad = b^2 - a^2 \end{array} \quad \rightarrow \quad \begin{array}{r} c^2 - 2cx = b^2 - a^2 \\ a^2 - b^2 + c^2 = 2cx \\ \dfrac{a^2 - b^2 + c^2}{2c} = x \qquad [3] \end{array}$$

The area of a triangle equals (1/2) × base × height. We know the base has length c units, so let's find h in terms of a, b and c. To start, we'll rearrange [2] and replace x with the expression in [3].

$$\begin{array}{r} h^2 + x^2 = a^2 \\ h^2 = a^2 - x^2 \\ h^2 = (a + x)(a - x) \end{array} \quad \rightarrow \quad \begin{array}{l} h^2 = \left[a + \dfrac{a^2 - b^2 + c^2}{2c}\right]\left[a - \dfrac{a^2 + b^2 - c^2}{2c}\right] \\ h^2 = \left[\dfrac{2ac + a^2 - b^2 + c^2}{2c}\right]\left[\dfrac{2ac - a^2 + b^2 - c^2}{2c}\right] \end{array}$$

Recall that $(a + c)^2 = a^2 + 2ac + c^2$ and $(a - c)^2 = a^2 - 2ac + c^2$, so we can substitute to get

$$h^2 = \left[\frac{a^2 + 2ac + c^2 - b^2}{2c}\right]\left[\frac{b^2 - (a^2 - 2ac + c^2)}{2c}\right] = \left[\frac{(a + c)^2 - b^2}{2c}\right]\left[\frac{b^2 - (a - c)^2}{2c}\right] =$$

$$\frac{(a + c - b)(a + c + b)(b - a + c)(b + a - c)}{4c^2} = \frac{(a + c + b)(-a + b + c)(a - b + c)(a + b - c)}{4c^2} \qquad [4]$$

The semiperimeter, s, of a triangle is half of the triangle's perimeter. For this particular triangle, $s = (a + b + c)/2$. This equation can be rewritten a number of different ways to find expressions equivalent to those in the numerator of [4].

Multiplying both sides of the equation by 2 gives us $2s = a + b + c$ **[6]**.

Subtracting $2a$ from each side of [6], we get

$$2s - 2a = a + b + c - 2a$$
$$2(s - a) = -a + b + c \qquad [7]$$

Subtracting $2b$ from each side of [6], we get

$$2s - 2b = a + b + c - 2b$$
$$2(s - b) = a - b + c \qquad [8]$$

Subtracting $2c$ from each side of [6], we get

$$2s - 2c = a + b + c - 2c$$
$$2(s - c) = a + b - c \qquad [9]$$

Now we can substitute these values for the corresponding expression in the numerator of [4]. Doing so yields the following equation:

$$h^2 = \frac{(2s)2(s-a)2(s-b)2(s-c)}{4c^2}$$

$$h^2 = \frac{16s(s-a)(s-b)(s-c)}{4c^2}$$

$$h^2 = \frac{4s(s-a)(s-b)(s-c)}{c^2}$$

Taking the square root of each side, we see that

$$h = \sqrt{\frac{4s(s-a)(s-b)(s-c)}{c^2}}$$

$$h = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{c}$$

$$ch = 2\sqrt{s(s-a)(s-b)(s-c)} \qquad \textbf{[10]}$$

As previously stated, the area of a triangle equals (1/2) × base × height, which is, in this case, (1/2)*ch*. Replacing *ch* with the expression in [10], we have shown that

$$\mathbf{A = \sqrt{s(s-a)(s-b)(s-c)}}$$