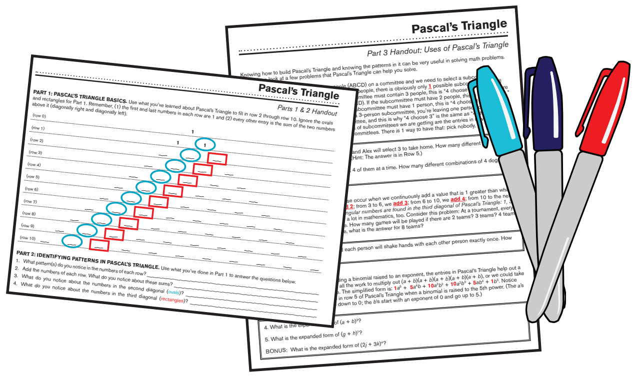
PASCAL'S TRIANGLE

Exploration Materials

Materials

- Part 1 PDF presentation
- Pascal's Triangle parts 1 & 2 handout
- Pascal's Triangle part 3 handout
- Answer key to parts 1 & 2 handout
- Solutions to part 3 handout
- Highlighters and/or markers (optional)
- Stopwatch (optional)



Exploration Activities

At the 2013 Raytheon MATHCOUNTS National Competition, 325 students, coaches and volunteers set a Guinness World Record for the fastest time to construct a human formation of the first 25 rows of Pascal's Triangle. Involving numbers ranging from 1 to 2,704,156, the task was completed in just 6 minutes, 16.57 seconds! Now what's so special about Pascal's Triangle? It's easy to create and it houses an amazing number of useful patterns! In this activity, students will learn the basics about Pascal's Triangle, identify some of the common patterns found within it, and use it to solve math problems.

Part 1: Pascal's Triangle Basics

Explain to students how to form the first 11 rows (row 0 through row 10) of Pascal's Triangle using the Part 1 PDF presentation. Students should use Part 1 of the first handout to complete the first 11 rows of Pascal's Triangle.

Part 2: Pascal's Showdown

Now that your students have learned the basics of Pascal's Triangle, have them practice with the entries through row 10 with a Pascal's Showdown! Have students race with each other to fill in a certain number of rows as quickly as possible, with all numbers correct. The more rows and the further inward along each row, the more difficult the numbers, so you can make this as simple or as challenging as you would like.

You can have students work on this contest at their desks by using a sheet of paper, but it may be more fun to have them play side by side at a chalkboard or dry-erase board or under a document camera. You can also have all the members of the math club race each other at the same time. For an added challenge, you can time the students and have them race against the clock, as well as each other!

Part 3: Identifying Patterns in Pascal's Triangle and Using It to Solve Problems

Students will use Part 2 of the first handout to identify patterns in Pascal's Triangle. They can also use highlighters or markers to note the patterns they find in their completed 11 rows of Pascal's Triangle. Next, students will use the Part 3 handout, which contains questions and guidance on how to use Pascal's Triangle to answer some math problems included on the handout. Refer to the solutions to Part 3 (below) for tips and suggestions for how to help students figure out how they can use Pascal's Triangle.

Solutions to Part 3 Handout

1. In math terms, this is "5 choose 3," and the answer will be in row 5: 1, 5, 10, 10, 5, 1. Alex has 1 way to pick 5 cookies, 5 ways to pick 4 cookies, **10 ways to pick 3 cookies**, and so on.
2. This is "7 choose 4," and the answer will be in row 7: 1, 7, 21, 35, 35, 21, 7, 1. There are **35 combinations of 4 dogs** Aaron could choose to walk first. This is the same number as the 35 combinations of 3 dogs he could choose to leave home first or "7 choose 3."
3. Our answers (starting with 2 people) will follow the numbers down the third diagonal (1, 3, 6, 10, 15, 21, 28, ...): 2 people → 1 handshake; 3 people → 3 handshakes; 4 people → 6 handshakes; 5 people → **10 handshakes**.
4. $(a + b)^3$ will use row 3: $1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3 = \mathbf{a^3 + 3a^2b + 3ab^2 + b^3}$
5. $(g + h)^7$ will use row 7: $1g^7h^0 + 7g^6h^1 + 21g^5h^2 + 35g^4h^3 + 35g^3h^4 + 21g^2h^5 + 7g^1h^6 + 1g^0h^7 = \mathbf{g^7 + 7g^6h + 21g^5h^2 + 35g^4h^3 + 35g^3h^4 + 21g^2h^5 + 7gh^6 + h^7}$

BONUS: $(2j + 3k)^4$ will use row 4: $1(2j)^4(3k)^0 + 4(2j)^3(3k)^1 + 6(2j)^2(3k)^2 + 4(2j)^1(3k)^3 + 1(2j)^0(3k)^4 = 1(16j^4) + 4(8j^3)(3k) + 6(4j^2)(9k^2) + 4(2j)(27k^3) + 1(81k^4) = \mathbf{16j^4 + 96j^3k + 216j^2k^2 + 216jk^3 + 81k^4}$

Do More with This Activity

Here are additional fun activities your club can do with Pascal's Triangle:

- Create Pascal's Triangle Hopscotch. Use sidewalk chalk to draw row 0 through row 10 of Pascal's Triangle, and then give students a short math problem that they could use Pascal's Triangle to solve (refer to the Part 3 handout for ideas). Students have to hop to the answer on the triangle.

- Go for your own Guinness Record attempt. Based on the number of students you have in your class, determine how many rows of Pascal's Triangle you could form, and then give each student a number within the rows. Have the students try to assemble themselves into Pascal's Triangle, with each student representing a number. Let the students try the challenge a couple times, striving to improve their time with each try. You can also have students work in teams and race against each other to do this as quickly as possible. If you do not have many students, place 1s on the floor and do not assign those numbers to students.

Pascal's Triangle

Parts 1 & 2 Handout

PART 1: PASCAL'S TRIANGLE BASICS. Use what you've learned about Pascal's Triangle to fill in row 2 through row 10. Ignore the ovals and rectangles for Part 1. Remember, (1) the first and last numbers in each row are 1 and (2) every other entry is the sum of the two numbers above it (diagonally right and diagonally left).

(row 0) 1

(row 1) 1 1

(row 2) ___ ___ ___

(row 3) ___ ___ ___ ___

(row 4) ___ ___ ___ ___ ___

(row 5) ___ ___ ___ ___ ___ ___

(row 6) ___ ___ ___ ___ ___ ___ ___

(row 7) ___ ___ ___ ___ ___ ___ ___ ___

(row 8) ___ ___ ___ ___ ___ ___ ___ ___ ___

(row 9) ___ ___ ___ ___ ___ ___ ___ ___ ___ ___

(row 10) ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___

PART 2: IDENTIFYING PATTERNS IN PASCAL'S TRIANGLE. Use what you've done in Part 1 to answer the questions below.

1. What pattern(s) do you notice in the numbers of each row? ______________________
2. Add the numbers of each row. What do you notice about these sums? ______________________
3. What do you notice about the numbers in the second diagonal (ovals)? ______________________
4. What do you notice about the numbers in the third diagonal (rectangles)? ______________________

Pascal's Triangle Answer Key

Parts 1 & 2 Handout

(row 0)	1
(row 1)	1 1
(row 2)	1 2 1
(row 3)	1 3 3 1
(row 4)	1 4 6 4 1
(row 5)	1 5 10 10 5 1
(row 6)	1 6 15 20 15 6 1
(row 7)	1 7 21 35 35 21 7 1
(row 8)	1 8 28 56 70 56 28 8 1
(row 9)	1 9 36 84 126 126 84 36 9 1
(row 10)	1 10 45 120 210 252 210 120 45 10 1

PART 2: IDENTIFYING PATTERNS IN PASCAL'S TRIANGLE. Use what you've done in Part 1 to answer the questions below.

1. What pattern(s) do you notice in the numbers of each row? ***The first and last numbers are the same; the second and second-to-last numbers are the same; the third and third-to-last numbers are the same; and so on.***
2. Add the numbers of each row. What do you notice about these sums? ***The sums are 1, 2, 4, 8, 16, 32, 64, 128, 256, 512 and 1024. The sum of each row is double the sum of the row above it; and the sums are all powers of 2. The sum of the numbers in row n is 2^n. For example, the sum of the numbers in row 10 is 2^{10} = 1024.***
3. What do you notice about the numbers in the second diagonal (ovals)? ***They are the positive integers and correspond to their row numbers… row 1 has a 1 in the second diagonal, row 2 has a 2, and so on.***
4. What do you notice about the numbers in the third diagonal (rectangles)? ***Starting with 1 in the third diagonal, the pattern is: add 2 (to get 3); add 3 (to get 6); add 4 (to get 10); add 5 (to get 15); and so on. These are the triangular numbers.***

Pascal's Triangle

Part 3 Handout: Uses of Pascal's Triangle

Knowing how to build Pascal's Triangle and knowing the patterns in it can be very useful in solving math problems. Let's take a look at a few problems that Pascal's Triangle can help you solve.

Combinations: Let's say there are 4 people (ABCD) on a committee and we need to select a subcommittee from this group. If the subcommittee must have 4 people, there is obviously only **1** possible subcommittee. It will contain everyone on the committee. If the subcommittee must contain 3 people, this is "4 choose 3," and there are **4** possible subcommittees (ABC, ABD, ACD, BCD). If the subcommittee must have 2 people, this is "4 choose 2," and there are **6** possible subcommittees. If the subcommittee must have 1 person, this is "4 choose 1," and there are **4** possible subcommittees. Notice that with a 3-person subcommittee, you're leaving one person out. Each person left out is a possible 1-person subcommittee, and this is why "4 choose 3" is the same as "4 choose 1." We are starting with 4 people, and the numbers of subcommittees we are getting are the entries in row 4: **1, 4, 6, 4, 1**! The last **1** is the number of 0-person subcommittees. There is 1 way to have that: pick nobody.

1. There are 5 different cookies in the bag, and Alex will select 3 to take home. How many different combinations of cookies could he select? (Hint: The answer is in Row 5.)

2. Aaron has 7 dogs, but he can only walk 4 of them at a time. How many different combinations of 4 dogs could he choose to walk first?

Triangular Numbers: 1, 3, 6, 10, … These occur when we continuously add a value that is 1 greater than what we added previously. From 1 to 3, we **add 2**; from 3 to 6, we **add 3**; from 6 to 10, we **add 4**; from 10 to the next number, we'll **add 5** to get 15. *These triangular numbers are found in the third diagonal of Pascal's Triangle: 1, 3, 6, 10, and so on.* These numbers appear a lot in mathematics, too. Consider this problem: At a tournament, every team plays every other team exactly once. How many games will be played if there are 2 teams? 3 teams? 4 teams? Answers: **1, 3, 6**. Using Pascal's Triangle, what is the answer for 8 teams?

3. There are 5 people at a party, and each person will shake hands with each other person exactly once. How many handshakes will take place?

Expanding Binomials: For expanding a binomial raised to an exponent, the entries in Pascal's Triangle help out a lot! Consider $(a + b)^5$. We could do all the work to multiply out $(a + b)(a + b)(a + b)(a + b)(a + b)$, or we could take a shortcut and use Pascal's Triangle. The simplified form is: $\mathbf{1}a^5 + \mathbf{5}a^4b + \mathbf{10}a^3b^2 + \mathbf{10}a^2b^3 + \mathbf{5}ab^4 + \mathbf{1}b^5$. Notice that the expansion uses the entries in row 5 of Pascal's Triangle when a binomial is raised to the 5th power. (The *a*'s start with an exponent of 5 and go down to 0; the *b*'s start with an exponent of 0 and go up to 5.)

4. What is the expanded form of $(a + b)^3$?

5. What is the expanded form of $(g + h)^7$?

BONUS: What is the expanded form of $(2j + 3k)^4$?