MARBLE CHALLENGE

Mathematical Exploration

Marble Challenge Strategy

The Marble Challenge game is derived from a 2015-2016 *MATHCOUNTS School Handbook* problem that was the focus of that year's winner of the Math Video Challenge. Solving the original problem and watching the video will help your club understand the strategy and will serve as a jumping-off point for the concept of modular arithmetic. Begin by letting students ponder the original problem, shown in the box. Then guide them with the following questions to dive deeper.

> *Ron and Martin are playing a game with a bowl containing 39 marbles. Each player takes turns removing 1, 2, 3 or 4 marbles from the bowl. The person who removes the last marble loses. If Ron takes the first turn to start the game, how many marbles should he remove to guarantee he is the winner?*
> *(2015-2016 MATHCOUNTS School Handbook #78)*

- ***What do you notice about the numbers of marbles that are can be removed in this game—1, 2, 3 or 4? Consider that the game is played by pairs of students, so think of pairs of numbers.***

  If you look at pairs that can be made with these numbers, you notice that there are two pairs (using all four numbers) that sum to the number 5. Namely, 1 + 4 and 2 + 3.

- ***How could using sums of 5 affect the game?***

  Since the game is played in pairs, then based on the opponent's previous choice, a player can always choose the number that will create a sum of 5. For example, if the opponent chooses 1, a player would choose 4.

- ***Can you now answer the problem's question about guaranteeing a win? Watch the video to see how some students explain the strategy: https://www.youtube.com/watch?v=A6VKI0Un2E0. Using the explanation they presented and your own understanding, can you now write a solution to the problem?***

  Solutions can vary, but here is the solution included in the 2015-2016 *MATHCOUNTS School Handbook*:

  Ron should take 3 marbles to leave Martin with 36. To explain this, we will work backward. At the end of the game, Ron wants to leave 1 marble so Martin will have to take the last marble. Thus, Martin will lose. To ensure that he can leave Martin with 1 marble, Ron should leave 6 marbles before that. By doing so, Ron can ensure that 5 marbles are removed between his turn and Martin's final turn. For example, if Martin takes 1 marble, Ron takes 4 marbles (or vice-versa). Or, if Martin takes 2, Ron takes 3 (or vice-versa). Therefore, as long as Ron leaves Martin with a number 1 more that a multiple of 5, Ron will completely control the game. Try it on an unsuspecting friend!

- ***We now have a strategy and know that the first player in a game with 39 marbles can guarantee a win with the right opening move. How does this apply to a game with 56 marbles? Would you want to go first or second? Why?***

  We know that 56 is 55 + 1, 1 more than a multiple of 5. This gives the advantage to Player B. No matter what the first player decides to take, the second player can use sums of 5 to control the game to the end and guarantee that the first player removes the last marble.

- ***Suppose we played the game again but changed the number of marbles allowed to be drawn. How would your strategy change if you were allowed to draw only 1, 2 or 3 marbles? What if you could draw 1, 2, 3, 4 or 5 marbles?***

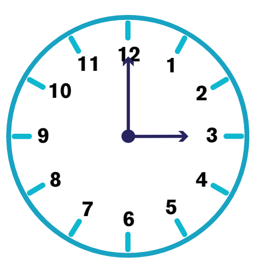
  Similarly to 1, 2, 3 and 4, let's look at the sets of numbers and determine a sum that can be consistently created when playing in pairs. For 1, 2, 3, we can always create a sum of 4 by adding 1 + 3 or 2 + 2. With 56 marbles, we notice this is already a multiple of 4. The first player should remove 3 marbles to obtain a number 1 more than a multiple of 4 and gain an advantage. For 1, 2, 3, 4, 5, we can make pairs that sum to 6. This means the first player should take 1 marble to obtain a total of 55, 1 more than a multiple of 6.

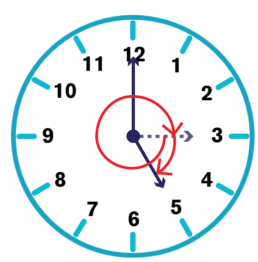
Modular Arithmetic

If your club members understood the Marble Challenge strategy and successfully applied it in game play, then they are doing modular arithmetic! In modular arithmetic, numbers "wrap around" once they reach a certain value—the modulus. The terms *modular arithmetic* and *modulus* might sound intimidating, but modular arithmetic is something all students are familiar with, whether they know it or not. The simplest example is a 12-hour clock.

- ***If it is 3 o'clock now, then what time will it be 14 hours later? (Ignore a.m. and p.m. for this exercise.)***

  A clock has a modulus of 12. This means the numbers "wrap around" after reaching 12. For this problem, 3 + 14 = 17, but we wouldn't say it is 17 o'clock. We would say it is 5 o'clock. This is because 17 = 12 × 1 + 5. In modular arithmetic, we can "remove" multiples of the modulus to get the final expressed value. In modular arithmetic notation, we would write this expression as $17 \equiv 5 \pmod{12}$. Let's apply this to our Marble Challenge game!





- ***What is 56 mod 5? We've already done this calculation but without modular arithmetic terminology or notation.***

  We know that 56 = 5 × 11 + 1. This can be expressed in modular arithmetic notation as $56 \equiv 1 \pmod{5}$.

- ***What is 39 mod 5?***

  Again, we know 39 = 5 × 7 + 4. This means $39 \equiv 4 \pmod{5}$. Feel free to continue having students practice modular arithmetic calculations. This game provides endless opportunities for calculations by changing the number of marbles or the numbers allowed to be drawn.