MATHCOUNTS®

2024 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide detailed solutions to the Sprint, Target and Team Rounds of the 2024 MATHCOUNTS State Competition. These solutions show creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions
with the MATHCOUNTS community.

Sprint 1
$2n + 10 = 48 \rightarrow 2n = 38 \rightarrow n = \mathbf{19}$.

Sprint 2
$\$0.75 \times \frac{32 \cancel{\text{egg}}}{10 \cancel{\text{egg}}} = \$\frac{3}{4} \times \frac{16}{5} = \$\frac{3\times 4 \cancel{\times 4}}{\cancel{4\times}5} = \$\frac{12}{5} = \$\mathbf{2.40}$.

Sprint 3
A straightforward equation for the mean, by definition, leads to $\frac{(3+(-4)+2+0+8+x)}{6} = 2 \rightarrow 12 = 3 + (-4) + 2 + 0 + 8 + x = 9 + x \rightarrow x = \mathbf{3}$. However, understanding the following logic could help with more complex problems in the future. The value 3 is 1 above the desired mean of 2; -4 is 6 below so 5 below net so far; 2 is right on; 0 is 2 below so 7 below net; 8 is 6 above so 1 below net; so x needs to be 1 above to balance out and end up with 0 net, so $x = 2 + 1 = \mathbf{3}$.

Sprint 4
4 divides y and 6 divides y implies $\text{lcm}(4, 6) = 12$ divides y. For the product of the digits to be 18 and for y to be even (so 4 can divide it), the digits must be 3 and 6, or 9 and 2. Out of 36 and 92, only **36** is a multiple of 12.

Sprint 5
Getting all 20 correct earns 100 points. Changing a correct answer to incorrect causes a loss of 8 points (loss of 5 points for not getting credit for spelling the word correctly and increasing the loss by 3 points for the incorrectness penalty). A score of 68 points means a loss of 32 points which corresponds to $32/8 = \mathbf{4}$ incorrect answers.

Sprint 6
The number of cents that a token is worth matches the number of grams in that token's mass. Therefore, 435 g of mass of tokens corresponds to 435¢ = $**4.35** value.

Sprint 7
For an outer circle of radius R and an inner circle of radius r, the area of the annulus is $\pi(R^2 - r^2)$. Area for $R = 16\text{ cm}/2 = 8\text{ cm}$ and $r = 5\text{ cm}$ is $\pi[(8\text{ cm})^2 - (5\text{ cm})^2] = \pi(64 - 25)\text{ cm}^2 = \mathbf{39\pi}\text{ cm}^2$.

Sprint 8
$\frac{12\,\cancel{¢}}{(3/4)\,\cancel{¢}} \times 32\text{ oz} = 12 \times \frac{4}{3} \times 32\text{ oz} = 4 \times 4 \times 32\text{ oz} = \mathbf{512}\text{ oz}$.

Sprint 9
Let a, b and c be ordered so that c is the greatest of the three to achieve maximum possible value; the median 5 is the middle value, so let $b = 5$. Then $\frac{a+5+c}{3} = 10 \rightarrow a + 5 + c = 30 \rightarrow c = 25 - a$, so c is maximized when a is minimized: $a \geq 1 \rightarrow c \leq \mathbf{24}$.

Sprint 10
Let Pr(M), Pr(T) and Pr(M∨T) be the probability of snow on Monday, on Tuesday and on either or both days, respectively. Then $1 - \Pr(\text{M})$ and $1 - \Pr(\text{T})$ are the probability of no snow on Monday and on Tuesday, respectively, so $(1 - \Pr(\text{M}))(1 - \Pr(\text{T})) = (1 - 0.5)(1 - 0.6) = 0.5 \times 0.4 = 0.2$ is the probability that it snows neither Monday nor Tuesday; otherwise, it snows at least one of the two days, so $\Pr(\text{M}\vee\text{T}) = 1 - 0.2 = 0.8 = \mathbf{80}\ \%$.

Sprint 11

The drop distance d and time t are related by $d = kt^2$, so letting d_2 and d_3 be drop distance after 2 s and 3 s, respectively, yields the ratio $\frac{d_2}{d_3} = \frac{\cancel{k}(2\ \cancel{s})^2}{\cancel{k}(3\ \cancel{s})^2} = \frac{4}{9}$, so $d_2 = \frac{4}{9}d_3 = \frac{4}{9}(144\text{ ft}) = 4(16\text{ ft}) = \mathbf{64}$ ft.

Sprint 12

Let the weights of the pumpkins be w_1, w_2, w_3, w_4, w_5 in increasing order of weight. The median of 5 sorted values must be $\#\frac{1+5}{2} = \#3$, so $w_3 = 73$ kg. Thus, 77 kg must go with w_4 or w_5 (or both). We have $w_1 \le w_2 \le 73\text{ kg} \le w_4 \le w_5$, and we wish to minimize $w_5 - w_1$. Let's see how small we can make w_5 and how big we can make w_1, consistent with the various constraints. [We must be careful though, because this does *not* necessarily mean that we can have the maximum possible value of w_1 simultaneously with the minimum possible value of w_5.] We must allow for the value of 77 kg, so we must have either $77\text{ kg} = w_4 \le w_5$ or $73\text{ kg} \le w_4 \le w_5 = 77$ kg. The less we make w_4, the greater we can make w_1 and w_2, and we must have such an offset in order to keep the mean at 73 kg. Thus, our latter option is more promising: $73\text{ kg} \le w_4 \le w_5 = 77$ kg, and we can try the extreme of making $w_4 = 73$ kg, so we are at $w_1 \le w_2 \le 73$ kg, $w_3 = w_4 = 73$ kg, $w_5 = 77$ kg. The 5 values having mean of 73 kg means: $0\text{ kg} = (w_1 - 73\text{ kg}) + (w_2 - 73\text{ kg}) + (w_3 - 73\text{ kg}) + (w_4 - 73\text{ kg}) + (w_5 - 73\text{ kg}) = (w_1 - 73\text{ kg}) + (w_2 - 73\text{ kg}) + 0\text{ kg} + 0\text{ kg} + 4\text{ kg} \rightarrow w_1 + w_2 = 73\text{ kg} + 73\text{ kg} - 4\text{ kg} = 142$ kg. Thus, the greatest that w_1 can be while keeping $w_1 \le w_2$ occurs with $w_1 = w_2 = (142\text{ kg})/2 = 71$ kg. Therefore, the least possible range is $w_5 - w_1 = 77\text{ kg} - 71\text{ kg} = \mathbf{6}$ kg.

Sprint 13

In 1 h, the minute hand moves 1 rotation, thus 360° while the hour hand moves $\frac{1}{12}$ rotation, thus $\frac{360°}{12} = 30°$. Therefore, the minute hand moves at an angular rate of $\frac{360°-30°}{1\text{ h}} = 330°/\text{h}$ relative to the hour hand. At 02:00, the minute hand is straight up at the 12, but the hour hand has moved to $2 \times 30° = 60°$ ahead of the minute hand. The minute hand needs to close the 60° gap to 0° and then move on to open the gap to 180° on the other side of the hour hand to be directly opposite. At a rate of 330°/h, this 240° shift will take $\frac{240^{\circ}}{330^{\circ}/\text{h}} = \frac{8\times30}{11\times30}\text{ h} = \mathbf{\frac{8}{11}}$ h.

Sprint 14

L→O: 4 options; O→DM: 3 options; DM→O: 3 options; O→L: 4 options.
The number of options for each of the 4 choices is independent of which option is picked for the other choices, so the total is the product of the options count at each choice: $4 \times 3 \times 3 \times 4 = \mathbf{144}$.

Sprint 15

Let x be the measure of the desired angle ∠DBC. By the isosceles triangle theorem, ∠ABD ≡ ∠ADB; let y be the measure of those two angles. As the supplementary angle of ∠ADB, ∠CDB has measure $180° - y$. The sum of the measures of the three angles of a triangle is 180°, so ∠DCB has measure $180° - (180° - y) - x = y - x$. We are given $30° = m\angle\text{ABC} - m\angle\text{ACB} = (x + y) - (y - x) = 2x$. Therefore, $x = \mathbf{15}°$.

Sprint 16

The two pairs of rectangular brackets enclose opposite values. Let $a = 2 - \left(\frac{5}{1024} * 1\right)$. Then the ultimate operation is $a * (-a)$. Let's evaluate this for arbitrary a as we might save time not plugging in numbers: $a * (-a) = \frac{a+(-a)+a(-a)}{a^2} = \frac{0-a^2}{a^2} = -1$ for all $a \ne 0$, which a satisfies, so the result is $\mathbf{-1}$.

Sprint 17

The first card dealt could have been A, B or C to have the ☯, with each being equally likely, thus probability $\frac{1}{3}$. The only cards satisfying the criteria for the second card are A [if applicable—not to be repeated from first choice], D, E and F. If A was the first card dealt, then only D, E and F out of the 5 remaining cards satisfy the desired criteria as the second card, for a probability of $\frac{3}{5}$; if either B or C was the first card dealt, then A, D, E and F out of the 5 remaining cards satisfy the desired criteria as the second card, for a probability of $\frac{4}{5}$. Combining the probabilities for each of the three possible first-card scenarios yields $\frac{1}{3} \times \frac{3}{5} + \frac{1}{3} \times \frac{4}{5} + \frac{1}{3} \times \frac{4}{5} = \frac{1}{3} \times \frac{3+4+4}{5} = \mathbf{\frac{11}{15}}$.

Sprint 18

Intersecting the x-axis means $y = 0$, which occurs at $-\sqrt{6}$ and $+\sqrt{6}$. For a quadratic equation, this means the equation is of the form $y = a\left(x + \sqrt{6}\right)\left(x - \sqrt{6}\right) = a(x^2 - 6)$. At $x = 0$, y is given to be 2, so $2 = a(0^2 - 6) = -6a \rightarrow a = -\frac{1}{3}$. Therefore, $y = -\frac{1}{3}(x^2 - 6) = -\frac{1}{3}x^2 + 2 = -\frac{1}{3}x^2 + 0x + 2$ $\rightarrow a + b + c = -\frac{1}{3} + 0 + 2 = \mathbf{\frac{5}{3}}$.

Sprint 19

$415 = a + ab + abc = a(1 + b + bc)$, so 415 is the product of two positive integers. The positive divisors of 415 are 1, 5, 83 and 415. With a, b, c being positive integers, each greater than or equal to 2, so $1 + b + bc \geq 7 > 5$, meaning $a = 5$ and $1 + b + bc = 83$. Thus, $82 = b + bc = b(1 + c)$. In turn, 82 has positive divisors 1, 2, 41 and 82. With $2 \leq b < 82$ and $3 \leq 1 + c < 82$, this means $b = 2$ and $1 + c = 41$. Therefore, $c = \mathbf{40}$.

Sprint 20

The surface area S of a right circular cone includes the base and the side and is given by $S = \pi r(r + l)$, where r is the radius of the circular base and l is the slant height, with $l^2 = r^2 + h^2$, where h is the vertical height of the cone. We are given that S has value $90\pi \text{ m}^2$, so $\pi r(r + l) = 90\pi \text{ m}^2$. Dividing both sides by πr yields $r + l = \frac{90 \text{ m}^2}{r} \rightarrow l = \frac{90 \text{ m}^2}{r} - r$. Square both sides and substitute $l^2 = r^2 + h^2$ to obtain $r^2 + h^2 = l^2 = \left(\frac{90 \text{ m}^2}{r} - r\right)^2 = \frac{8100 \text{ m}^4}{r^2} - 180 \text{ m}^2 + r^2$. Equating the left end and right end of this string of equations and subtracting r^2 from both sides yields $h^2 = \frac{8100 \text{ m}^4}{r^2} - 180 \text{ m}^2$. We are given $h = 12$ m, so $144 \text{ m}^2 = \frac{8100 \text{ m}^4}{r^2} - 180 \text{ m}^2 \rightarrow 324 \text{ m}^2 = \frac{8100 \text{ m}^4}{r^2} \rightarrow r^2 = \frac{8100 \text{ m}^4}{324 \text{ m}^2} = \frac{\cancel{81} \times 100}{\cancel{81} \times 4} \text{ m}^2$.

Therefore, $r^2 = 25 \text{ m}^2$, and the volume $V = \frac{\pi r^2 h}{3} = \frac{\pi(25 \text{ m}^2)(12 \text{ m})}{3} = \mathbf{100\pi} \text{ m}^3$.

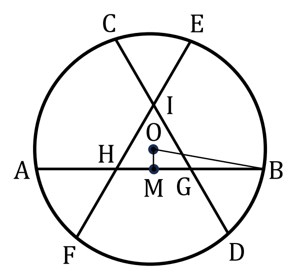
Sprint 21

$9^6 - 1 = (9^3 + 1)(9^3 - 1)$; $9^5 - 9^3 - 9^2 + 1 = (9^2 - 1)(9^3 - 1) = 80(9^3 - 1)$. Thus, $9^3 - 1$ is a factor of both values, so it cannot contribute a prime factor of the first expression without dividing the second expression as well. Therefore, our answer must come from $9^3 + 1 = 730 = 2 \times 5 \times 73$, of which 73 is the only prime factor of the first expression that is not a factor of the second expression, so the answer is **73**.

Sprint 22

In this case, P(failure) is easier to calculate than P(success), since "failure" can only happen in two distinct ways: (1) no dime heads or (2) 1 dime head and no nickel heads. [We will then remember, P(success) = 1 − P(failure).] Let's first determine that for the two dimes, P(0 heads) = (1/2)(1/2) = 1/4, P(2 heads) = 1/4 and P(1 head) = 1/2. Now, we've already calculated the probability of the first failure scenario – flipping no heads with the dimes – to be ¼. Note that only the dime tosses are of concern in this failure scenario. The second failure scenario is a bit more involved. It requires no nickel heads [probability is (1/2)(1/2)(1/2) = 1/8] and exactly one dime head, which we've already calculated to be 1/2, so the probability of our second failure scenario is (1/8)(1/2) = (1/16). Thus, the probability of failure is (1/4) + (1/16) = 5/16, and the probability of success is 1 − 5/16 = **11/16**.

Sprint 23



Each segment portion is $\frac{1}{3}$ of 3 cm, thus 1 cm. Each side of $\triangle$GHI is 1 cm long, so the triangle is equilateral. Point O, the center of the circle and the centroid of $\triangle$GHI. Let M be the midpoint of both $\overline{\text{AB}}$ and $\overline{\text{GH}}$. The altitude MI is $\frac{\sqrt{3}}{2}$ cm; the distance from the centroid to the base is 1/3 of the altitude, so $\text{MO} = \frac{\sqrt{3}}{6}$ cm; $\text{MB} = \frac{3}{2}$ cm. Therefore, $\overline{\text{OB}}$ is the radius of circle O and the hypotenuse of $\triangle$BOM, with length $r = \sqrt{\frac{3}{36} + \frac{9}{4}}$ cm $= \sqrt{\frac{84}{36}}$ cm $= \sqrt{\frac{7}{3}}$ cm, so circle O encloses area $\pi r^2 = \boldsymbol{\frac{7\pi}{3}}$ cm^2.

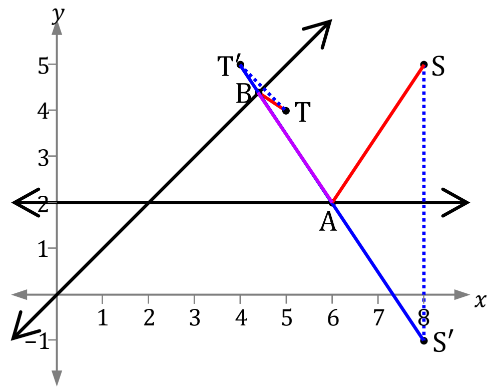
Sprint 24

There are $6^3 = 216$ outcomes of dice rolls (a, b, c), with $v = |a - b| + |b - c| + |c - a|$:

Pattern	Frequency f	Value v	fv
(a, a, a) for $1 \le a \le 6$	6	$0 + 0 + 0 = 0$	0
$(a, a, a + 1)$, $(a, a + 1, a + 1)$, and perm.: $1 \le a \le 5$	$(3 + 3) \times 5 = 30$	$0 + 1 + 1 = 2$	60
$(a, a, a + 2)$, $(a, a + 2, a + 2)$, and perm.: $1 \le a \le 4$	$(3 + 3) \times 4 = 24$	$0 + 2 + 2 = 4$	96
$(a, a, a + 3)$, $(a, a + 3, a + 3)$, and perm.: $1 \le a \le 3$	$(3 + 3) \times 3 = 18$	$0 + 3 + 3 = 6$	108
$(a, a, a + 4)$, $(a, a + 4, a + 4)$, and perm.: $1 \le a \le 2$	$(3 + 3) \times 2 = 12$	$0 + 4 + 4 = 8$	96
(1, 1, 6), (1, 6, 6), and permutations	$3 + 3 = 6$	$0 + 5 + 5 = 10$	60
$(a, a + 1, a + 2)$ and permutations: $1 \le a \le 4$	$6 \times 4 = 24$	$1 + 1 + 2 = 4$	96
$(a, a + 1, a + 3)$, $(a, a + 2, a + 3)$, and perm.: $1 \le a \le 3$	$(6 + 6) \times 3 = 36$	$1 + 2 + 3 = 6$	216
$(a, a + 1, a + 4)$, $(a, a + 3, a + 4)$, and permutations for $1 \le a \le 2$	$(6 + 6) \times 2 = 24$	$1 + 3 + 4 = 8$	192
(1, 2, 6), (1, 5, 6), and permutations	$6 + 6 = 12$	$1 + 4 + 5 = 10$	120
$(a, a + 2, a + 4)$ and permutations: $1 \le a \le 2$	$6 \times 2 = 12$	$2 + 2 + 4 = 8$	96
(1, 3, 6), (1, 4, 6), and permutations	$6 + 6 = 12$	$2 + 3 + 5 = 10$	120
Total	216	—	1260

The expected value is the Total fv divided by the Total f: $1260/216 = \mathbf{35/6}$.

Sprint 25



Let S = (8, 5) be the path's starting point and T = (5, 4) be its termination point. Let S′ = (8, −1) be the point of reflection of S about the line $y = 2$, and T′ = (4, 5) be the point of reflection of T about the line $y = x$. The red path S→A→B→T is a potential path of the beetle; each segment is a straight segment in order to minimize path length. The solid blue path S′→A→B→T′ is a counterpart "virtual" path involving projected points. (Note: The $\overline{\text{AB}}$ segment is shared by both paths, so it is shaded purple.) The dotted blue lines show the indicated reflections. △SAS′ is isosceles because it is the union of two congruent right triangles (sharing a common leg on the line $y = 2$, with the other pair of legs being congruent based on the reflection), so $\overline{\text{SA}}$ and $\overline{\text{S}'\text{A}}$ are congruent. Similarly, △TBT′ is isosceles, with $\overline{\text{TB}}$ and $\overline{\text{T}'\text{B}}$ being congruent. This means that regardless where A is on $y = 2$ and where B is on $y = x$, the two paths have the same total length. It is easier to minimize the length of path S′→A→B→T′ than the length of path S→A→B→T. The length of a path between two points is minimized if the path is straight. Thus, place A and B where line segment $\overline{\text{S}'\text{T}'}$ intersects $y = 2$ and $y = x$, respectively, namely A = (6, 2) and B = (4.4; 4.4). Therefore, the optimum path length is the distance between S′ and T′, $\sqrt{(8-4)^2 + (-1-5)^2} = \sqrt{4^2 + 6^2} = \sqrt{52} = \mathbf{2\sqrt{13}}$.

[NOTE: The reference to the path of a beetle makes for cute but unrealistic amusement. However, this problem has real-world application for aiming a laser beam reflecting off of two mirrors to be able to hit a designated point. The bisector of ∠SAB is perpendicular to line $y = 2$ and the bisector of ∠ABT is perpendicular to line $y = x$, matching with angle of reflection equals angle of incidence.]

Sprint 26

$165 = xy(x+z)(y+z) = xyxy + xyyz + xyxz + xzyz = (xy+xz)(xy+yz);$ [1]
$88 = xz(x+y)(z+y) = xzxz + xzyz + xzxy + xyyz = (xz+yz)(xz+xy);$ [2]
$120 = yz(z+x)(y+x) = yzyz + yzxz + yzxy + xyxz = (yz+xy)(yz+xz).$ [3]

On the right end of equations [1] to [3], each factor $(xy+xz)$, $(xy+yz)$, $(xz+yz)$ occurs twice in total (perhaps with terms switched), and nothing else, so the square root of the product of all three equations yields:

$\sqrt{165 \times 88 \times 120} = \sqrt{3 \times 5 \times 11 \times 8 \times 11 \times 3 \times 5 \times 8} = 3 \times 5 \times 8 \times 11 =$
$1320 = (xy+xz)(xy+yz)(xz+yz).$ [4]

We can now successively divide [4] by each of [1] to [3] to yield [5] through [7], respectively:
$8 = xz + yz;$ [5] $15 = xy + yz;$ [6] $11 = xy + xz.$ [7]

Each of xy, xz and yz occurs two times in total in equations [5] to [7], and nothing else occurs, so half the sum of all three equations yields: $xy + xz + yz = (8 + 15 + 11)/2 = 17.$ [8]

We can now successively subtract [8] minus each of [5] to [7] to yield [9] through [11], respectively:
$xy = 9;$ [9] $xz = 2;$ [10] $yz = 6.$ [11]

Multiplying [9] by [10] and dividing by [11] yields $x^2 = 9 \times 2/6 = 3.$ [12]
Multiplying [9] by [11] and dividing by [10] yields $y^2 = 9 \times 6/2 = 27.$ [13]
Multiplying [10] by [11] and dividing by [9] yields $z^2 = 2 \times 6/9 = \frac{4}{3}.$ [14]

Adding equations [12] to [14] yields $x^2 + y^2 + z^2 = 3 + 27 + \frac{4}{3} = \frac{9+81+4}{3} = \mathbf{\frac{94}{3}}$.

Sprint 27

Let S be the center of the shaded circle and M be the midpoint of segment $\overline{AB}$. With segment $\overline{AB}$ being a chord of both circle O and circle S, it follows that $\overline{OM}$ and $\overline{SM}$ are both perpendicular to $\overline{AB}$ at its midpoint M, and therefore O, S and M are collinear. Create an xy-coordinate system with origin at O and positive x-axis along ray $\overrightarrow{OM}$. We are given that circle O encloses an area of $625\pi\ \text{cm}^2$, so the radius is $\sqrt{625\ \text{cm}^2} = 25$ cm, with $\overline{OA}$ being such a radius. We are given $AB = 30$ cm, so M being the midpoint means $AM = 15$ cm. $\triangle AOM$ is a 3-4-5 right triangle with scale factor 5 cm, so $OM = 20$ cm. We are given $CD = 16$ cm, so $OC = OD - CD = 25\ \text{cm} - 16\ \text{cm} = 9$ cm. Thus, the xy-coordinates, as number of centimeters, of key points are: $O = (0, 0)$; $M = (20, 0)$; $A = (20, 15)$. Let $P = (x, 0)$ for yet-to-be-determined value of x. C is 9/25 as far from O as A is and on the opposite side of O, so $C = (-7.2; -5.4)$. Let r be the radius of circle S. Then $SA = SB = SC = r$. We can use $\overline{SC}$ and either $\overline{SA}$ or $\overline{SB}$ to determine r (there is a strong interdependence between $\overline{SA}$ and $\overline{SB}$, so we can use only one of those.
$SA^2 = (x - 20\ \text{cm})^2 + (15\ \text{cm})^2 = x^2 - (40\ \text{cm})x + 625\ \text{cm}^2$. [The $625\ \text{cm}^2$ comes from 3-4-5 $\triangle$ with scale factor 5 cm and squaring hypotenuse.]
$SC^2 = (x + 7.2\ \text{cm})^2 + (5.4\ \text{cm})^2 = x^2 + (14.4\ \text{cm})x + 81\ \text{cm}^2$. [The $81\ \text{cm}^2$ comes from 3-4-5 $\triangle$ with scale factor 1.8 cm and squaring hypotenuse.]
Setting these two equal yields $x^2 - (40\ \text{cm})x + 625\ \text{cm}^2 = x^2 + (14.4\ \text{cm})x + 81\ \text{cm}^2$, so $(54.4\ \text{cm})x = 544\ \text{cm}^2 \rightarrow x = 10$ cm. Therefore, $r^2 = SA^2 = (10\ \text{cm} - 20\ \text{cm})^2 + (15\ \text{cm})^2 = 100\ \text{cm}^2 + 225\ \text{cm}^2 = 325\ \text{cm}^2$ and the area enclosed by circle S is $\pi r^2 = \mathbf{325\pi}\ \text{cm}^2$.

Sprint 28

$\sqrt{\frac{(n+3)!+(n-1)!}{(n-1)!}} - 1 = \sqrt{n(n+1)(n+2)(n+3)+1} - 1$ for any positive integer n.

$$\begin{aligned} n(n+1)(n+2)(n+3)+1 &= [n(n+3)][(n+1)(n+2)] + 1 \\ &= \left[\left(n+\tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\right)^2\right]\left[\left(n+\tfrac{3}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2\right] + 1 \\ &= \left(n+\tfrac{3}{2}\right)^4 - \tfrac{5}{2}\left(n+\tfrac{3}{2}\right)^2 + \tfrac{25}{16} = \left[\left(n+\tfrac{3}{2}\right)^2 - \tfrac{5}{4}\right]^2 = (n^2+3n+1)^2. \end{aligned}$$

Thus, $\sqrt{\frac{(n+3)!+(n-1)!}{(n-1)!}} - 1 = \sqrt{(n^2+3n+1)^2} - 1 = n^2 + 3n + 1 - 1 = n^2 + 3n = n(n+3)$, because $n^2 + 3n + 1 > 0$ for $n > 0$. For the question at hand, $n = 2020$, so we need the prime factorization of 2020×2023. Now, $2020 = 2^2 \times 5 \times 101$; $2023 = 7 \times 289 = 7 \times 17^2$, which consolidates to: $2020 \times 2023 = 2^2 \times 5^1 \times 7^1 \times 17^2 \times 101^1$, with $(2+1)(1+1)(1+1)(2+1)(1+1) = 3^2 \times 2^3 = \mathbf{72}$ positive integer factors.

Sprint 29

The pair of equations $x^4 + y^4 = 48$ (Lamé curve, hyperellipse) and $xy = 2$ (equilateral hyperbola) have some nice symmetries, in particular swapping the roles of x and y has no impact on the equations so a symmetry axis of reflection $y = x$, and changing the sign of both x and y together has no impact on the equations so a symmetry through the origin. This means that if (x, y) is a solution, then $(-x, -y)$, (y, x) and $(-y, -x)$ are also solutions. As a consequence, we also have a symmetry axis of reflection $y = -x$ and a 2-fold rotation symmetry, not 4-fold as some people sometimes erroneously assume. Let's rewrite the hyperbola equation as $y = \frac{2}{x}$ and substitute that in the hyperellipse equation: $x^4 + \frac{16}{x^4} = 48$. Multiply through by x^4 and rearrange: $x^8 - 48x^4 + 16 = 0$. This equation is quadratic in x^4, so solve using quadratic equation: $x^4 = 24 \pm \sqrt{560}$ (not reducing intermediate calculations). Let's try determining the square root of this, to find x^2:

Let $x^2 = \sqrt{a} + \sqrt{b}$. Then $x^4 = \left(\sqrt{a} + \sqrt{b}\right)^2 = a + b + 2\sqrt{ab} = (a + b) + \sqrt{4ab}$, so we need a and b such that $a + b = 24$ and $ab = 140$, which is satisfied by $a = 14, b = 10$, so $x^2 = \sqrt{14} \pm \sqrt{10}$. Thus, (x, y) is either of $\pm\left(\sqrt{\sqrt{14} + \sqrt{10}}, \sqrt{\sqrt{14} - \sqrt{10}}\right)$ or $\pm\left(\sqrt{\sqrt{14} - \sqrt{10}}, \sqrt{\sqrt{14} + \sqrt{10}}\right)$, where in both cases, either the $+$ or the $-$ may be applied to both components of the ordered pair (no mixing of one sign to one component and the other sign to the other component). These four solutions form the vertices of a rectangle, with two vertices in Quadrant I and two in Quadrant III. First find the distance between the two Quadrant1 points: Note that the difference of the x-components is the negative of the difference in y, so their squares will match—just double the square of the x-component differences: $\sqrt{2\left(\sqrt{\sqrt{14} + \sqrt{10}} - \sqrt{\sqrt{14} - \sqrt{10}}\right)^2} =$ $\sqrt{2\left[\left(\sqrt{14} + \sqrt{10}\right) + \left(\sqrt{14} - \sqrt{10}\right) - 2\sqrt{\left(\sqrt{14} + \sqrt{10}\right)\left(\sqrt{14} - \sqrt{10}\right)}\right]} = \sqrt{2\left(2\sqrt{14} - 2 \times 2\right)} =$ $\sqrt{4\sqrt{14} - 8}$. Now let's similarly handle a perpendicular side of the rectangle. Let's take as endpoints $\left(\sqrt{\sqrt{14} + \sqrt{10}}; \sqrt{\sqrt{14} - \sqrt{10}}\right)$ and $\left(-\sqrt{\sqrt{14} - \sqrt{10}}; -\sqrt{\sqrt{14} + \sqrt{10}}\right)$—the negative of each of these would also work, but a pairing of the form $(x; y)$ with $(-x; -y)$ yields a diagonal, not a side, by symmetry. Again the square of the differences match for the x- and y-components, so apply factor 2 to x-term inside square root: $\sqrt{2\left(\sqrt{\sqrt{14} + \sqrt{10}} + \sqrt{\sqrt{14} - \sqrt{10}}\right)^2} =$ $\sqrt{2\left[\left(\sqrt{14} + \sqrt{10}\right) + \left(\sqrt{14} - \sqrt{10}\right) + 2\sqrt{\left(\sqrt{14} + \sqrt{10}\right)\left(\sqrt{14} - \sqrt{10}\right)}\right]} = \sqrt{2\left(2\sqrt{14} + 2 \times 2\right)} =$ $\sqrt{4\sqrt{14} + 8}$. Therefore, the area enclosed by the rectangle is the product of the two adjacent sides: $\sqrt{4\sqrt{14} - 8}\sqrt{4\sqrt{14} + 8} = \sqrt{\left(4\sqrt{14}\right)^2 - 8^2} = \sqrt{16 \times 14 - 64} = \sqrt{160} = \mathbf{4\sqrt{10}}$.

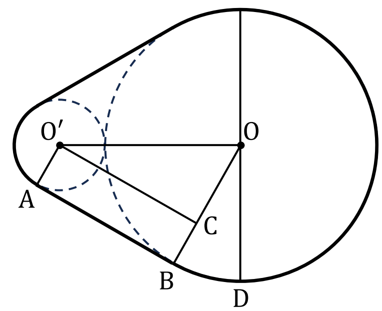
<u>Sprint 30</u>

The given expression $103^4 + 101^4 + 2^4$ is of the form $m^4 + n^4 + (m + n)^4$, with $m = 101$ and $n = 2$.
$m^4 + n^4 + (m + n)^4 = m^4 + n^4 + m^4 + 4m^3n + 6m^2n^2 + 4mn^3 + n^4$
$= 2m^4 + 4m^3n + 6m^2n^2 + 4mn^3 + 2n^4 = 2(m^4 + 2m^3n + 3m^2n^2 + 2mn^3 + n^4) = 2(m^2 + mn + n^2)^2$.
[The coefficient pattern 1-2-3-2-1 corresponds to 111^2, which involves the square of a trinomial.]
Thus, the expression given in the question equals $2(101^2 + 101 \times 2 + 2^2)^2 = 2(10\,201 + 202 + 4)^2$ $= 2(10\,407)^2$. We immediately see 2 as a prime factor. Any prime factor of $10\,407^2$ must be a factor of 10 407 itself. The sum of the digits of 10 407 is 12, which is divisible by 3, so 3^2 divides $10\,407^2$: $10\,407 = 3 \times 3469$. We have found two prime factors, 2 and 3, neither of which divides 3469, so based on the given assumption, which we will trust [and it is, in fact, true], that there are three [distinct] prime factors, the 3469 that remains must be either prime or a power of a prime. To be a perfect square, number being squared would need to be between 50 and 60, closer to 60, and end in 3 or 7—57^2 is 3249, so no square (nor fourth power, …). To be a perfect cube, the number being cubed would need to be between 10 and 20 and end in 9, but 19^3 is way too big. For higher powers, the prime factor must be less than 10, we already have 2 and 3, 5 clearly does not work (3469 not ending in 0 or 5), and trying 7 yields a remainder of 1. Thus, 3469 is the remaining prime. Of the three choices, 2, 3 and 3469, the greatest is **3469**.

Target 1

$j + \quad b = 16$ oz [1] — Given.
$3j + 3b = 48$ oz [2] — Multiplying [1] by 3.
$j + 3b = 24$ oz [3] — Given.
$2j \qquad = 24$ oz [4] — Subtract [2] minus [3]. Therefore, $j = 24$ oz/$2 = \mathbf{12}$ oz.

Target 2



Segment $\overline{OO'}$ connects the centers of externally tangent circles O and O′, so its length is (18 cm + 6 cm)/2 = 12 cm. Let A and B be the lower points where the band transitions between straight (segment $\overline{AB}$) and circular (wrapping around the logs) arcs of O′ and O, respectively. Construct a diameter of circle O perpendicular to $\overline{OO'}$ and let D be the lower intersection point of the diameter with circle O. Segments $\overline{O'A}$ and $\overline{OB}$ are perpendicular to $\overline{AB}$ (a line tangent to a circle is perpendicular to the radius of the circle at the point of tangency). Construct a segment from O′ to $\overline{OB}$ and parallel to $\overline{AB}$ (and, therefore, perpendicular to $\overline{OB}$) with point of intersection being C. CB = O′A = 3 cm. Thus, OC = OB − CB = 9 cm − 3 cm = 6 cm. OO′C is a right triangle, the length of whose leg $\overline{OC}$ is 1/2 that of hypotenuse OO′, so $m\angle OO'C = 30°$. That means $AB = O'C = OC \times \sqrt{3} = 6\sqrt{3}$ cm. There is a like straight segment in the upper half of the figure. $\angle BOD \cong \angle OO'C$, thus measuring 30°. By symmetry, there is a like arc in the top half plus a full semicircle to the right of the line $\overleftrightarrow{OD}$. Thus, the band is in contact with the large log for $180° + 2 \times 30° = 240°$ of arc, which is $\frac{2}{3}$ of the circumference of circle O, which is $\frac{2}{3} \times 2\pi(9\text{ cm}) = 12\pi$ cm. The remaining 120° of circular arc of band contacting a log is on the left side of circle O′, which means $\frac{1}{3}$ of the circumference of circle O′, which is $\frac{1}{3} \times 2\pi(3\text{ cm}) = 2\pi$ cm. Therefore, the length of the band is $\left(12\pi + 2\pi + 2 \times 6\sqrt{3}\right)\text{ cm} = \left(14\pi + 12\sqrt{3}\right)\text{ cm} = 64.7669 \ldots$ cm, which rounds to **65** cm.

Target 3

$c = (84 - 63)/3 = 7$ cars. Having 1 additional person per car ride back means 7 more people rode back, so 7 less people rode the bus back, so $63 - 7 = \mathbf{56}$ bus riders on the return trip.

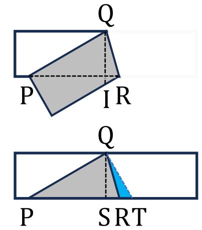
Target 4

Let n be number of attendees and s be the sum of their ages.
$n = 1 + 2 + 11 + 13 + 12 + 9 + 6 + 2 + 7 + 2 + 5 + 1 + 2 + 2 = 75.$
$s = (1 \times 16 + 2 \times 17 + 11 \times 18 + 13 \times 19 + 12 \times 20 + 9 \times 21 + 6 \times 23 + 2 \times 24 + 7 \times 28 + 2 \times 30 + 5 \times 35 + 1 \times 36 + 2 \times 38 + 2 \times 40)$ yr
$= 16 + 34 + 198 + 247 + 240 + 189 + 138 + 48 + 196 + 60 + 175 + 36 + 76 + 80 = 1733$ yr.
The mean age value = (1733 yr)/75 = 23.106… yr. The median age is age of #(1 + 75)/2 = #38 attendee. The first 4 histogram blocks have 27 values, and the first 5 have 39 values, so #38 is in the fifth block, which is age 20 yr. The absolute difference is 23.106… yr − 20 yr = 3.106… yr, which rounds to one decimal place as **3.1** yr.

Target 5

This is a geometric series with 7 terms (for 7 days of the week), initial term 5 and common ratio 3. Therefore, the sum is $\frac{5(3^7-1)}{3-1} = \frac{5\times 2186}{2} = \mathbf{5465}$.

Target 6



The upper figure is Step 3 in the question, plus labels P, Q and R introduced early from Step 4, plus point I being added as the point where a line is dropped from Q perpendicular to the top edge of the unshaded part of the strip down to the bottom edge of the shaded part of the strip; In the lower figure, S is the intersection of $\overline{QI}$ and $\overline{PR}$. If you cut both layers along $\overline{QI}$ and unfold the right piece, you get triangle QST. The gray-shaded QSR is from the topside of the lower folded piece and the blue-shaded QRT is from the underside of the upper folded piece, with the unfolding process moving $\overline{QI}$ from the upper piece to become $\overline{QT}$ in the unfolded strip. By construction, $\overline{QR}$ bisects ∠SQT, $\overline{QT} \perp \overline{QP}$ and the angle between $\overline{QP}$ and the upper edge of the strip in the lower figure is congruent to ∠SQT. Let the measure of ∠SQR and ∠RQT be m. Then ∠SQT, ∠QPR and the angle between the upper edge of the strip are congruent and have measure $2m$. Angles RQT and PQR are complementary, so the measure of ∠PQR is $90° - m$, and the measure of ∠PRQ is $180° - (2m + 90° - m) = 90° - m$. Thus, ∠PQR and ∠PRQ, so △QPR is isosceles. We were given that the area of △QPR is 36 cm², we constructed $\overline{QS} \perp \overline{PR}$ and QS = 6 (the width of the strip), so 36 = (1/2)(PR)(QS) = (1/2)(PR)(6) and PR = 12 cm. Thus, for isosceles triangle △QPR, PQ = PR = 12 cm. In △PQS, PQ = 12 cm is the hypotenuse and QS = 6 cm is a leg of the right triangle, so we have a 30°-60°-90° triangle with the other leg PS being $6\sqrt{3}$ cm, SR = $(12 - 6\sqrt{3})$ cm. Therefore, by Pythagoras QR = $\sqrt{(6\text{ cm})^2 + \left[(12 - 6\sqrt{3})\text{ cm}\right]^2} = \sqrt{36 + 144 + 108 - 144\sqrt{3}}$ cm = $\sqrt{288 - 144\sqrt{3}}$ cm = 6.211 657… cm, which rounds to **6.21** cm.

Target 7

AD ≤ AB + BC + CD = 4 cm + 3 cm + 7 cm = 14 cm. [Note: Even though we do not know which of the three lengths goes with which of the three segments, the sum of the three lengths does not depend on the order.] The three choices for AD are 14 cm and up. Therefore, AD ≤ 14 cm and AD ≥ 14 cm, which together require AD = **14** cm.

Target 8

Let's first determine the probability of the underdog team winning a round in 3, 4 or 5 games, along with the total, to determine the total probability of the favored team to win a round:

Underdog to win in 3: Win all 3 games: $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$.

Underdog to win in 4: Win any 2 games and lose 1 of first 3, then win 4th: $\frac{3!}{2!\,1!}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{27} \times 2$.

Underdog to win in 5: Win any 2 games and lose 2 of first 4, then win 5th: $\frac{4!}{2!\,2!}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{1}{27} \times \frac{8}{3}$.

Underdog to win altogether: $\frac{1}{27}\left(1 + 2 + \frac{8}{3}\right) = \frac{1}{27} \times \frac{17}{3} = \frac{17}{81}$.

Therefore, the probability for the favored team to win is $1 - \frac{17}{81} = \frac{64}{81}$.

Connerstown is underdog to win their semifinal round, so probability 17/81 to make it to final. In the other semifinal round, Appleton is favored for probability 64/81 to make it to final, where they would be favored over Connerstown, with the latter having probability of 17/81 to win final; Belleville is the underdog or probability 17/81 to make it to final, where they would be underdog to Connerstown, who would have probability of 64/81 to win final.

Putting it all together, Connerstown has probability of $\frac{17}{81} \times \frac{64}{81} \times \frac{17}{81} + \frac{17}{81} \times \frac{17}{81} \times \frac{64}{81} = 2 \times \frac{17^2 \times 64}{81^3}$ $= \frac{36\,992}{531\,441} = 0.069\,606\ldots = 6.9606\ldots\,\%$, which rounds to **6.96** %.

Team 1

$19\frac{17}{25} = 19.68$. Therefore, $19.68\ \frac{\cancel{\text{in}}}{\cancel{\text{s}}} \times 2\ \cancel{\text{h}} \times \frac{3600\ \cancel{\text{s}}}{1\ \cancel{\text{h}}} \times \frac{1\ \cancel{\text{ft}}}{12\ \cancel{\text{in}}} \times \frac{1\ \text{mi}}{5280\ \cancel{\text{ft}}} = 2.236\ 36\ldots$ mi, which rounds to **2.24** mi.

Team 2

Buying 12 carrots means 5/6 of them, or 10 carrots, are regular price, and the other 2 are discounted: $\frac{260\ ¢-2\times 15\ ¢}{12-2} = \frac{230\ ¢}{10} = \mathbf{23}$ ¢ for the regular price per carrot.

Team 3

There are 4 one-digit primes (2, 3, 5, 7) With six digits left, perhaps we can get 3 two-digit primes. However, multi-digit primes must end in 1, 3, 7 or 9, but two of those are used, so only 2 more primes can possibly work. We can have 2, 3, 5, 7, 41, 89 for a total count of **6** prime terms.

Team 4

Our goal is to find the first band of 6 values we can make, since continuing to add a 6 to each of them will produce the next 6 values over and over again. The following scores are possible using each point value on its own:

6: 6, 12, 18, 24, 30, 36, … 10: 10, 20, 30, 40, … 15: 15, 30, 45, 60, …

Until we use at least one 15, we will not be able to get any odd scores totals. So, let's start checking odd numbers greater than 15 until we can get at least 3 consecutive odd numbers. The next possible odd number is 15 + 6 = 21; then 15 + 10 = 25; then 15 + 6 + 6 = 27; then 15 + 6 + 10 = 31; then 15 + 6 + 6 + 6 = 33; then 15 + 10 + 10 = 35. We've found three consecutive odd values; if we can construct 30, 32 and 34, we will have found our band of 6 values: 30 = 15 + 15; 32 = 10 + 10 + 6 + 6; 34 = 6 + 6 + 6 + 6 + 10. So, {30, 31, 32, 33, 34, 35} are each possible, meaning each value greater than 35 also is possible. We were not able to make **29**, so that is the greatest unachievable integer.

Team 5

Create a coordinate system with origin at point B and x-axis including segment $\overline{\text{AB}}$. Thus, A = (14, 0); B = (0, 0), C = (0, 14). The locus of all points (x, y) at distance 15 units from A are given by $(x-14)^2 + y^2 = (15)^2$. The locus of all points (x, y) at distance 13 units from B are given by $x^2 + y^2 = (13)^2$. Expanding the first equation yields: $x^2 - 28x + 196 + y^2 = 225$, so [1] $x^2 - 28x + y^2 = 29$ and [2] $x^2 + y^2 = 169$. Performing [2] minus [1] yields $28x = 140$. Thus, $x = \frac{140}{28} = 5$ cm. With $y > 0$, $y^2 = (13)^2 - (5)^2 = 144$ implies $y = 12$. Therefore, P = (5, 12) and CP $= \sqrt{(5-0)^2 + (12-14)^2} = \sqrt{25+4} = \sqrt{\mathbf{29}}$.

Team 6

Let $t_i = ar^{i-1}$ for $1 \le i \le 20$ be the ith term of the geometric sequence. Then:
$32 = t_4 t_8 t_{12} t_{16} t_{20} = ar^3 ar^7 ar^{11} ar^{15} ar^{19} = a^5 r^{55} = (ar^{11})^5 = (t_{12})^5$, so $t_{12} = 2$;
$4096 = t_5 t_{10} t_{15} t_{20} = ar^4 ar^9 ar^{14} ar^{19} = a^4 r^{46} = (ar^{11})^4 r^2 = (t_{12})^4 r^2 = 2^4 r^2 = 16r^2$, so $r^2 = 256$ and r is either $+16$ or -16;
$t_3 t_6 t_9 t_{12} t_{15} t_{18} = a^6 r^{2+5+8+11+14+17} = a^6 r^{57} = (ar^{11})^6 r^{-9} = (t_{12})^6 r^{-9} = 2^6(\pm 16)^{-9} = \pm 2^6 2^{-36} = \pm 2^{-30} = 2^k$, so $r = +16$ and $k = \mathbf{-30}$.

Team 7

Our desired prime must be greater than 31. In order to maximize this value while keeping the mean at 31, all other primes must be less than 31. Let's say we end up with n prime values included in our mean: If the prime number 2 is included in the mean, there are $n - 1$ odd values [all primes other than 2 being odd] and 1 even value, so the odd-even parity of our sum is the same as the parity of $n - 1$ but the sum must be $31n$ to have a mean of 31, so the sum has the parity of n, which contradicts the sum having the parity of $n - 1$; therefore, 2 must be excluded from consideration for the mean. Including versus excluding 31 has exactly the same outcome, so let's never use it—keep things as simple as possible. The greatest prime must be $31n$ minus the sum of all the odd primes less than 31 that we are using—but that difference does not always have the required primality. The ideal situation would be that including all 9 odd primes less than 31 would work, but, alas: $31 \times 10 - (3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 + 29) = 183$ is divisible by 3, so not prime. Thus, we must remove one or more. We would like to remove as few values as possible and those that we remove be as close to 31 as possible—these two goals can conflict. Let's first try removing only 29 (the closest to 31, so having least impact on the mean):
$31 \times 9 - (3 + 5 + 7 + 11 + 13 + 17 + 19 + 23)$ removes 31 and adds 29 [a net decrease of 2] to our first sum of 183 to yield 181, which is prime. We cannot do better than that, so the answer is **181**.

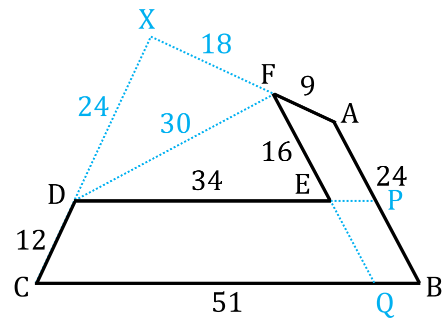
Team 8

The probability of getting the two-headed coin is 1/10; when it does happen, the probability of getting 10 heads is 1, so the probability of both happening is $\frac{1}{10} \times 1 = \frac{1}{10}$. The probability of getting a standard heads/tails coin is 9/10; when it does happen, the probability of getting 10 heads is $(1/2)^{10} = 1/1024$, so the probability of both happening is $\frac{9}{10} \times \frac{1}{1024} = \frac{9}{10\,240}$. These two scenarios are disjoint, so the probability of getting 10 heads for the given conditions is the sum $\frac{1}{10} + \frac{9}{10\,240} = \frac{1024}{10\,240} + \frac{9}{10\,240} = \frac{1033}{10\,240}$. The conditional probability that an event A happens whenever an event B happens is given by $\Pr(A \text{ and } B)/\Pr(\mathrm{B})$, which for this problem is $\frac{1024\cancel{/10\,240}}{1033\cancel{/10\,240}} = \mathbf{\frac{1024}{1033}}$.

Team 9

The relevant powers of 3 are: $3^0 = 1$; $3^1 = 3$; $3^2 = 9$; $3^3 = 27$; $3^4 = 81$; $3^5 = 243$; $3^6 = 729$. The terms being separated in value by a factor of 2 or more means that distinct contents in the sums yield distinct sum values, so we need not be concerned with overlapping sum values. There are 7 terms from which to choose potentially 2, 4 or 6 to form the sum. The total count of distinct combinations permitted is ${}_7C_2 + {}_7C_4 + {}_7C_6 = \frac{7!}{2!\,5!} + \frac{7!}{4!\,3!} + \frac{7!}{6!\,1!} = 21 + 35 + 7 = 63$. However, some of these violate the criterion of the sum not exceeding 1000. Exceeding 1000 requires at least 4 terms, with two of them being 729 and 243, with these latter two summing to 972, leaving 28 to spare. The only way to exceed 1000 is if at least one of 81 and 27 must be included. If 81 is included, we are definitely over: 81 must be paired with either one other term, with 4 to choose from, or 81 must be combined with three other terms, thus leaving out any one lower term, of which there are 4 to chose from; this means we have so far rejected 8 of the 63 to leave 55 to consider. The last case to consider is 81 is excluded and 27 is included, with one (any one of 9, 3 or 1) or three other terms (all of 9, 3 and 1) also included—a total of 4 possibilities. The only way for 27 to work is to be paired with 1, so the other 3 possibilities fail, leaving **52** that work.

Team 10



Extend segments $\overline{AF}$ and $\overline{CD}$ so that they intersect (X). The sum of the measures of the interior angles of quadrilateral FEDX is 360°; we are given that the sum of all of them except ∠DXF is 270°, so ∠DXF is a right angle. Extend segment $\overline{DE}$ to intersect $\overline{AB}$ (P); extend segment $\overline{FE}$ to intersect $\overline{BC}$ (Q). Because $\overline{AB} \parallel \overline{FQ}$ and $\overline{DP} \parallel \overline{CB}$: ∠ DEF and ∠CBA are corresponding and, thus, congruent, and ∠XDE and ∠XCB are corresponding and congruent; ∠EFX and ∠BAX are corresponding and congruent. Thus, ∠XDE ≅ ∠XCB, ∠DEF ≅ ∠CBA, ∠EFX ≅ ∠BAX and ∠FXD ≅ ∠AXC. So, quadrilaterals XDEF and XCBA are similar. Sides DE and CB are corresponding with linear scale ratio 34 : 51 = 2/3. Thus, $\frac{2}{3} = \frac{XD}{XC} = \frac{XD}{XD+12}$, so XD = 24; $\frac{2}{3} = \frac{FX}{AX} = \frac{FX}{FX+9}$, so FX = 18. Because ∠X is a right angle, and 18 and 24 are leg lengths of a 3-4-5 right triangle with scale factor 6, so hypotenuse FD has length $6 \times 5 = 30$, and △DFX encloses area $18 \times 24/2 = 216$. Now, △EFD is a 8-15-17 right triangle with scale factor 2 and encloses area $16 \times 30/2 = 240$. The total area enclosed by quadrilateral XDEF is $216 + 240 = 456$. Thus, the area enclosed by quadrilateral XCBA is $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ the area enclosed by XDEF, and the area enclosed by the hexagon is the difference in area of the two quadrilaterals, therefore $\left(\frac{9}{4} - 1\right) 456 = \frac{5 \times 456}{4} = \mathbf{570}$.