2023 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide detailed solutions to the Sprint, Target and Team Rounds of the 2023 MATHCOUNTS State Competition. These solutions show creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions with the MATHCOUNTS community.

Sprint 1
$4.7/2 = 2.35 \approx \mathbf{2}$.

Sprint 2
$\frac{10^3}{3^4-1} = \frac{1000}{81-1} = \frac{100\cancel{0}}{8\cancel{0}} = \mathbf{12.5}$.

Sprint 3
$3t < 260$, so $t < 260/3$. So, the greatest integer value for t is $(260/3) - 1 = (86\frac{2}{3}) - 1 = 85\frac{2}{3} \approx \mathbf{86}$.

Sprint 4
$m\angle A + m\angle B + m\angle C = 180°$, so $m\angle C = 180° - 40° - 85° = \mathbf{55}°$.

Sprint 5
$6 = \frac{4+8+6+9+n}{5}$, so $n = 5 \times 6 - (4+8+6+9) = 30 - 27 = \mathbf{3}$.

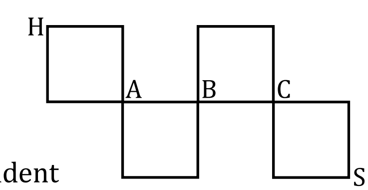
Sprint 6
$2021 + 3 \times 6 = 2021 + 18 = \mathbf{2039}$.

Sprint 7
For rectangles of fixed perimeter, the maximum area occurs for the square, so a side is $\frac{36\text{ mm}}{4} =$ 9 mm, and the area is $(9\text{ mm})^2 = \mathbf{81}\text{ mm}^2$.

Sprint 8
$200\,\cancel{\text{gj}} \times \frac{5\,\cancel{\text{Tsp}}}{15\,\cancel{\text{gj}}} \times \frac{1\text{ c}}{16\,\cancel{\text{Tsp}}} = \frac{200}{3\times16}$. So, $\frac{25}{3\times2} = \frac{25}{6} = \mathbf{4\frac{1}{6}}$ cups of flour are required to make 200 gulab jamun.

Sprint 9
$(a+b)^2 = (a-b)^2 + 4ab = 3^2 + 4 \times 40 = 169$, with $a > b > 0$, so $a + b = \sqrt{169} = \mathbf{13}$.

Sprint 10
$\frac{12°-6°}{360°} \times 24\,\cancel{\text{h}} \times 60\,\frac{\text{min}}{\cancel{\text{h}}} = \frac{1}{\cancel{60}} \times 24 \times \cancel{60}\text{ min} = \mathbf{24}$ minutes.

Sprint 11


Going from H to S requires passing through A, B, and C. There are 2 equidistant paths from H to A; there is only 1 minimum distance path, the direct one, from A to B and likewise from B to C; there are 2 equidistant paths from C to S. The choice of path from C to S is independent of the choice from H to A, so a total of $2 \times 1 \times 1 \times 2 = \mathbf{4}$ qualifying routes.

Sprint 12
At a minimum of 2 points per basket, 8 baskets is a minimum of 16 points, with 1 added point for each three-point basket. Lisa has $21 - 16 = 5$ added points, so she made **5** three-point baskets.

Sprint 13

Because the answer is to be in terms of whole hours, let's see what happens in 1 hour = 60 minutes. Now, 60 minutes = 15×4 minutes, so 15 snowballs are made, but 4 pairs of snowballs melt, so a net of $15 - (4 \times 2) = 7$ snowballs are produced per hour. So, gaining 21 snowballs requires **3** hours.

Sprint 14
The sum of 7 consecutive odd integers being 217 means the middle value is $217/7 = 31$. The least term is 3 odd integers below 31, thus $31 - 3 \times 2 = \mathbf{25}$.

Sprint 15
$2 \otimes (3 \otimes (-1)) = 2 \otimes (3^2 + (-1) - 3(3)) = 2 \otimes (-1) = 2^2 + (-1) - 3(2) = \mathbf{-3}$.

Sprint 16
The area of triangle PTQ is the area of rectangle PQRS minus the area of pentagon PTQRS, thus $(5\text{ cm})(12\text{ cm}) - 36\text{ cm}^2 = 24\text{ cm}^2$. That is the product of half the base PQ, or 6 cm, times the height MT, where M is the midpoint of $\overline{\text{PQ}}$. Thus, MT is 4 cm. $\text{PM} = \text{PQ}/2 = 6$ cm. The hypotenuse is then $\text{PT} = \sqrt{\text{PM}^2 + \text{MT}^2} = \sqrt{(6\text{ cm})^2 + (4\text{ cm})^2} = \sqrt{52}\text{ cm} = \mathbf{2\sqrt{13}}\text{ cm}$.

Sprint 17
During the 20 s run, L is running 0.4 m/s faster than A, which results in a $20\text{ s} \times 0.4\text{ m/s} = 8\text{ m}$ greater distance for L than A, so A must start **8** m ahead.

Sprint 18
The quadrilateral can be decomposed into two right triangles, one with legs x and $x + 4$ and the other with legs 5 and 5, and sharing a common hypotenuse. Thus, $x^2 + (x + 4)^2 = 5^2 + 5^2$, so $2x^2 + 8x + 16 = 50$, reducing to $x^2 + 4x - 17 = 0$, which by the quadratic formula has one positive solution: $x = \frac{-4+\sqrt{16+4(17)}}{2} = \mathbf{-2 + \sqrt{21}}$ OR $\mathbf{\sqrt{21} - 2}$.

Sprint 19
We want the fraction of the board outside the circle halfway between the 1 in and the 5 in circles, that is the 3 in circle, but inside the circle halfway between the 5 in and the 7 in circles, that is the 6 in circle. The area of all the relevant circles is proportional to the square of the radius of the circles. Total area proportional to $7^2 = 49$; region of interest is between 3 and 6, for area proportional to $6^2 - 3^2 = 27$. Desired probability is the ratio of the latter to the former, $\mathbf{\frac{27}{49}}$.

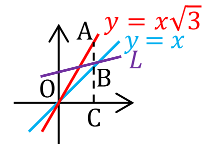
Sprint 20
$\frac{x+y}{2} = 7$ so $x + y = 14$; $\sqrt{xy} = 5$ so $xy = 25$ and $x^2 + y^2 = (x + y)^2 - 2xy = 14^2 - 2(25) = \mathbf{146}$.

Sprint 21
$44 = 4 \times 11$, so the number must be divisible by both 4 and 11. The last two digits taken together as the value 52 is divisible by 4, no matter the value of A, B, C. Divisibility by 11 means the alternating difference-sum of the digits is divisible by 11, so $11 \mid (2 - 5 + 0 - \text{C} + \text{B} - \text{A} + 2 - 5 + 3)$, thus $11 \mid (-\text{A} + \text{B} - \text{C} - 3)$, or, equivalently, $11 \mid (\text{A} - \text{B} + \text{C} + 3)$. Because A, B, C are digits, each is in the range 0 ... 9, and $0 - 9 + 0 + 3 = -6 \le \text{A} - \text{B} + \text{C} + 3 \le 9 - 0 + 9 + 3 = 21$, the only such multiples of 11 are 0 and 11. Thus, $\text{B} = \text{A} + \text{C} + 3$, in which case $0 \le \text{A} + \text{C} \le 6$, or $\text{B} = \text{A} + \text{C} - 8$, in which case $8 \le \text{A} + \text{C} \le 17$. The former case is the usual problem of the number of distinct sums [order important] of two nonnegative integers, which sum is less than or equal to n, being $(n + 1)(n + 2)/2$, thus 28 for $n = 6$. The latter case is harder with constraints on each digit: (A; 0; C) goes from (0; 0; 8) to (8; 0; 0) for 9 choices, (A; 1; C) goes from (0; 1; 9) to (9; 1; 0) for 10 choices, (A; 2; C) goes from (1; 2; 9) to (9; 2; 1) for 9 choices, (A; 3; C) goes from (2; 3; 9) to (9; 3; 2) for 8 choices, and so on to (A; 9; C) goes from (8; 9; 9) to (9; 9; 8) for 2 choices, making a total of $9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 = 63$ choices. All totaled is $28 + 63 = \mathbf{91}$ distinct values.

Sprint 22

The A on the front must go with the 2 on back, of which there is probability 1/4 of happening. The A on back must go with the 7 or 9, not the B, on front, of which there is probability 2/3 of happening. The two events are independent, so the overall probability is $\frac{1}{4} \times \frac{2}{3} = \mathbf{\frac{1}{6}}$.

Sprint 23

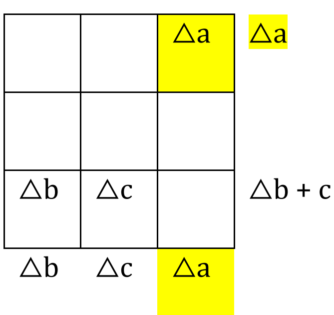


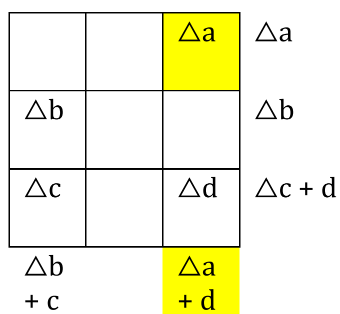
The vertical dashed line and the x-axis combine with the blue line to form a rt △ scaled as 1-√3-2 and with the red line to form a rt △ scaled as 1-1-√2 . Thus, $m\angle AOC = 60°$ and $m\angle BOC = 45°$. The purple line L has a slope that is 31° off (up or down) from the 45° slope of the blue line; in order to maximize the slope difference between the 60° of the red line and purple line L, the slope of L must be below that of the blue line, thus $45° - 31° = 14°$. Thus, the slope difference between the red line and the purple line is $60° - 14° = \mathbf{46}°$.

Sprint 24

Since the sum of each row or column is different, at best, we can pick any one row or any one column to have the "correct" sum and change no elements in that row or column. Each of the remaining 5 rows and columns has a different and, thus, "incorrect" sum, which needs to be adjusted with at least one element in each needing changed.

If we look at the 3 × 3 array independently of the given sums, we can determine that changing 1, 2 or 3 entries will not be sufficient. Changing 1 entry can at most affect two sums, changing 2 entries can at most affect four sums. Changing 3 entries can affect five sums, but two of these five sums will have to be adjusted by the same amount and since we know the values are different, this cannot result in them becoming equal. However, we can change five sums by different amounts by adjusting **4** entries.





We can verify this by choosing a row or column as "correct"—I will pick column 2 so the "correct" sum is 15. We can fix row 1 by changing only the 1 to 0 or the 8 to 7—let's do the latter. We can fix row 2 by changing only the 6 to 9 or the second 3 to 6—let's do the former. Because all six sums are different, the fixes to the first two rows have not fixed any columns, so we must change the 3 to 5 and the 9 to 5, fixing both columns and, in the process, also the third row. Thus, the array can be fixed by adjusting **4** entries.

Sprint 25

For $76 \le p \le 99$, there are 3 factors of p in 300!, 2 factors of p in 200!, and 1 factor of p in 100!, so 3 factors of p in the denominator cancel all 3 factors of p in the numerator. For $67 \le p \le 75$, there are 4 factors of p in the numerator but still only 3 in the denominator, so there remains a factor of p in the quotient. The largest prime in that range is **73**.

Sprint 26

How many distinct ordered pairs $(R - L; U - D)$ are there for nonnegative integers R, L, U, and D satisfying $R + L + U + D = 100$? For $U = D = 0$, $(R; L)$ is $(R; 100 - R)$, which ranges from $(0; 100)$ to $(100; 0)$, yielding 101 distinct locations for $(2R - 100; 0)$. Increasing U to 1 (keeping $D = 0$) has $(R; L)$ being $(R; 99 - R)$, which ranges from $(0; 99)$ to $(99; 0)$, yielding 100 distinct locations for $(2R - 99; 1)$. Each time we increase U by 1, we decrease the number of satisfying points by 1. This continues until $U = 100$, with only 1 satisfying point, namely $(0; 100)$ with $R = L = D = 0$. A like pattern results keeping $U = 0$ is successively incrementing D by 1. Thus, the total location count is $1 + 2 + 3 + \cdots + 99 + 100 + 101 + 100 + 99 + \cdots + 3 + 2 + 1 =$ ~~$2 \times$~~ $100(101)$~~$/2$~~ $+ 101 =$ **10 201**.

Sprint 27

$\overline{\text{RS}} \cong \overline{\text{RQ}}$ given; $\overline{\text{OS}} \cong \overline{\text{OQ}}$ as both are radii of the semicircle. Thus, OQRS is a kite whose area is 1/2 the product of the diagonal lengths: diagonal OR = 3 as a radius; diagonal $\overline{\text{QS}}$ is a leg of rt $\triangle$SPQ [inscribed in a semicircle with hypotenuse being a diameter of a semicircle], whose length is $\sqrt{6^2 - 2^2} = \sqrt{32} = 4\sqrt{2}$. Thus, the area of kite OQRS is $3 \times 4\sqrt{2}/2 = 6\sqrt{2}$. The area of $\triangle$OPQ is $\sqrt{3^2 - 1^2} \times 2/2 = 2\sqrt{2}$. The total shaded area is $6\sqrt{2} + 2\sqrt{2} = \mathbf{8\sqrt{2}}$.

Sprint 28

① $2xy + 16 = x^2$ ② $2xy + 9 = 4y^2$.

① + ②: $4xy + 25 = x^2 + 4y^2$, so $25 = x^2 - 4xy + 4y^2 = (x - 2y)^2$, so $x - 2y = \pm 5$.

① − ②: $7 = x^2 - 4y^2 = (x + 2y)(x - 2y) = (x + 2y)(\pm 5)$

Case $x - 2y = +5$: $5(x + 2y) = 7$, so $(x + 2y) = 1.4$ and $4y = -3.6$, violating $y > 0$. ☒

Case $x - 2y = -5$: $-5(x + 2y) = 7$, so $(x + 2y) = -1.4$ and $4y = 3.6$, $y = 0.9$, $x = -3.2$. Thus, $x + y = \mathbf{-2.3}$.

Sprint 29

Let $a = 3 + \sqrt{12}$ and $b = \sqrt[4]{1728}$. We are seeking $\left(\sqrt{a + b} + \sqrt{a - b}\right)^4$, which expands as:

$[a^2 + 2ab + b^2] + \left[4(a + b)\sqrt{a^2 - b^2}\right] + [6(a^2 - b^2)] + \left[4(a - b)\sqrt{a^2 - b^2}\right] + [a^2 - 2ab + b^2]$

$= 8a^2 - 4b^2 + 8a\sqrt{a^2 - b^2} = 8\left(21 + 6\sqrt{12}\right) - 4\sqrt{1728} + 8\left(3 + \sqrt{12}\right)\sqrt{21 - 6\sqrt{12}}$

$= 168$ ~~$+ 48\sqrt{12} - 48\sqrt{12}$~~ $+ 8\left(3 + \sqrt{12}\right)\left(\sqrt{12} - 3\right) = 168 + 8(3) = \mathbf{192}$.

Sprint 30

$V_\text{i} = \frac{15+5\sqrt{5}}{12} \times 2^3 = 10 + \frac{10}{3}\sqrt{5} = V_\text{c} + V_\text{d} = b + \frac{15+7\sqrt{5}}{4}a$. In order for a and b to be rational numbers, the equations must work separately for the $\sqrt{5}$ terms and for the rational terms. Thus, $\frac{10}{3}\sqrt{5} = \frac{7a}{4}\sqrt{5}$ and $a = \frac{4\times 10}{3\times 7} = \frac{40}{21}$. Then $10 = b + \frac{15}{4}a = b + \frac{15}{4} \times \frac{40}{21} = b + \frac{50}{7}$, so $b = 10 - \frac{50}{7} = \frac{20}{7}$. Thus, $a + b = \frac{40}{21} + \frac{20}{7} = \frac{40+60}{21} = \mathbf{\frac{100}{21}}$.

Target 1
There are 4 operations, each of which can be put independently in the two blanks, so $4^2 = 16$ distinct expressions. Here are the results:

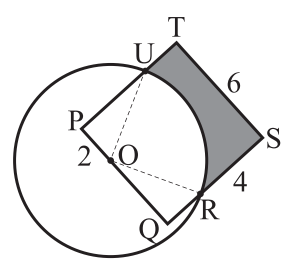
$4+3+3=10$	$4-3+3=4$	$4\times3+3=15$	$4\div3+3=13/3$
$4+3-3=4$	$4-3-3=-2$	$4\times3-3=9$	$4\div3-3=-2/3$
$4+3\times3=13$	$4-3\times3=-5$	$4\times3\times3=36$	$4\div3\times3=4$
$4+3\div3=5$	$4-3\div3=3$	$4\times3\div3=4$	$4\div3\div3=4/9$

Duplicates are highlighted in red. As shown, there are **13** distinct result values.

Target 2
We start with the point $\mathrm{P}(m, n)$. Reflecting P across the x-axis yields $\mathrm{Q}(m, -n)$. Reflecting Q across the y-axis yields $\mathrm{R}(-m, -n)$. Thus, PQR is a right triangle with base $\mathrm{QR} = 2|m|$ and height $\mathrm{PQ} = 2|n|$, so the area is $2|mn| = 80$. Thus, $|mn| = 40$. The usual rule to minimize a sum for a fixed product of two values is to choose the two values to be as close to equal as possible while satisfying constraints [like values being integers]; however, that rule applies to a straight sum of two positive values, but here we have absolute value of a sum and potentially negative values. Here we need m and n to be as close to each other as possible in magnitude but opposite in sign. The closest integers yielding a product of 40 are 5 and 8, whose sum is 13, but flipping the sign of the 5 yields a sum of **3**.

Target 3
The year 2023 has 365 days, which is 52 weeks + 1 day. Thus, the first day of the year occurs a 53rd time in the year while all other days of the week occur 52 times. Because the first day of the year is on Sunday, there are **53** Sundays.

Target 4

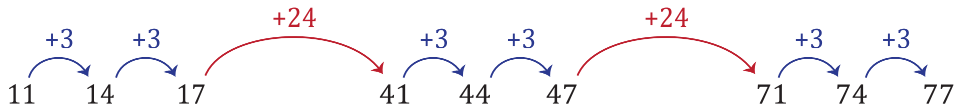


Let U be the intersection of $\overline{\mathrm{PT}}$ and circle O. The unshaded region of the square can be partitioned into sector ROU and congruent right △s OUP and ORQ as shown. Because the corresponding sides $\overline{\mathrm{UP}}$ and $\overline{\mathrm{OQ}}$ are perpendicular as are $\overline{\mathrm{PO}}$ and $\overline{\mathrm{QR}}$, we have $\overline{\mathrm{OR}} \perp \overline{\mathrm{OU}}$. So, angle ROU is a right angle, and sector ROU constitutes $1/4$ of circle O. Since UP = 4 and OP = 2, if follows that circle O has radius $\sqrt{4^2+2^2} = \sqrt{20}$ cm. Thus, sector ROU has area $\frac{\pi(\sqrt{20})^2}{4} = 5\pi$ cm^2. The combined area of the two triangles is $2 \times \frac{1}{2} \times 4 \times 2 = 8$ cm^2. Finally, the square has area $6^2 = 36$ cm^2. Therefore, the shaded region has area $36 - 8 - 5\pi = (\mathbf{28 - 5\pi})$ cm^2 or $(\mathbf{-5\pi + 28})$ cm^2.

Target 5
The mean is $(24 + x)/5$. If $x \le 3$, then the median is 3, so $4.8 + x/5 = 3$, and $x = 5(3 - 4.8) = -9$. If $x \ge 9$, then the median is 9, so $4.8 + x/5 = 9$, and $x = 5(9 - 4.8) = 21$. Otherwise, the median is x, so $4.8 + x/5 = x$, and $x = (4.8)5/4 = 6$. The sum of these values of x is $-9 + 21 + 6 =$ **18**.

Target 6

The $3 \times 3 = 9$ possible two-digit numbers whose digits are drawn from the set $\{1, 4, 7\}$ are shown in ascending order along with the differences from one number to the next.



So, we have the following cases for "The number $\underline{B}$ is the arithmetic mean of $\underline{A}$ and $\underline{C}$.":

Case 1: $A = B = C$ can occur in $\underline{9}$ ways.
Case 2: $B - A = C - B = 3$ is true for 3 sets of A, B, and C, each with 2 orders of A and C, for $3 \times 2 = \underline{6}$ ways.
Case 3: $B - A = C - B = 27$ is true for 1 set with 2 orders of A and C, for $1 \times 2 = \underline{2}$ ways.
Case 4: $B - A = C - B = 30$ is true for 3 sets, each with 2 orders of A and C, for $3 \times 2 = \underline{6}$ ways.
Case 5: $B - A = C - B = 33$ is true for 1 set with 2 orders of A and C, for $1 \times 2 = \underline{2}$ ways.

That's a total of $9 + 6 + 2 + 6 + 2 =$ **25** ways.

Target 7

The given right triangle is an 8-15-17 Pythagorean triple with a scale factor of 5 meters. For all sides to be an integer number of meters, the 2023 meters must be an integer multiple of one of the sides, that is, 2023 must be divisible by at least one of 8, 15, or 17. We know that 2023 is not a multiple of 8 since it is odd. We also know that 2023 is not a multiple of 15 because it does not have a units digit of 5 or 0. So, 17 needs to work, and $2023 = 17 \times 119$. Thus, we have an 8-15-17 right triangle with a scale factor of 119 m, making the area $\frac{(8\times119)(15\times119)}{2} = 60 \times (119)^2 = \mathbf{849,660}\ \text{m}^2$.

Target 8

We are to maximize $a(1 + r + r^2 + r^3 + r^4 + r^5) = a\frac{1-r^6}{1-r}$ for positive integers $a < 2023$ and $ar^5 < 2023$. It is easier to have the big end at a to process the size constraint, so $0 < r < 1$. For all terms to be integers, r must be rational, so of the form of a common fraction p/q. For ar^5 to be an integer, a must be a multiple of q^5, and we want the largest such multiple. $0 < r < 1$ requires $q > 1$. The largest terms are achieved for $p = q - 1$, so $r = \frac{q-1}{q} = 1 - \frac{1}{q}$. The sum yields $a\frac{1-r^6}{1-r} = aq(1 - r^6)$. For $q = 2$ the largest allowed multiple of $2^5 = 32$ is 2016, and $r = 1/2$ to yield $(2016)(2)\left(1 - \left(\frac{1}{2}\right)^6\right) = 4032 \times \frac{63}{64} = 3969$. For $q = 3$ the largest allowed multiple of $3^5 = 243$ is 1944, and $r = 2/3$ to yield $(1944)(3)\left(1 - \left(\frac{2}{3}\right)^6\right) = 5832 \times \frac{665}{729} = 5320$. For $q = 4$ the largest allowed multiple of $4^5 = 1024$ is 1024, and $r = 3/4$ to yield $(1024)(4)\left(1 - \left(\frac{3}{4}\right)^6\right) = 4096 \times \frac{3367}{4096} = 3367$. For $q = 5$ we need a positive multiple of $5^5 = 3125$ that is less than 2023—no such value exists, likewise for $q > 5$, so we have all candidates. Thus, max(3969; 5320; 3367) = **5320**.

Team 1

◖ + ◖ + ◖ = 1.5, so ◖ = 0.5; 13 = ◖ + ◖ + ■ = 1 + ■, so ■ = 12; 18 = ■ + ◆ = 12 + ◆, so ◆ = **6**.

Team 2

The number of 7th graders with blue eyes is $97 - (38 + 28) = 31$. The number of 7th graders with green eyes is $59 - (20 + 15) = 24$. Therefore, the number of 7th graders with brown eyes is $176 - (31 + 24) = \mathbf{121}$.

Team 3

Since 22 = 11 × 2, and that 2 is already handled by checking divisibility by 4, we need only check for divisibility by 4 and 11. For 4, 60 + B must be divisible by 4, so B must be 0, 4, or 8. For 11, $\mathrm{B} - 6 + \mathrm{A} - 9 + 3 - 8 = \mathrm{A} + \mathrm{B} - 20$ must be divisible by 11, so A + B must be 9 and $|\mathrm{A} - \mathrm{B}| = |(\mathrm{A} + \mathrm{B}) - 2\mathrm{B}| = |9 - 2\mathrm{B}|$. The greatest value achieved out of B being 0, 4, or 8 is for $\mathrm{B} = 0$, $\mathrm{A} = 9$, and $|\mathrm{A} - \mathrm{B}| = |9 - 0| = \mathbf{9}$.

Team 4

Let R be the radius of each ball. To be packed tightly, the box must have length and height $4R$ and width $2R$. Thus, the box encloses volume $(4R)(4R)(2R) = 32R^3$. The four balls take up volume $4\left(\frac{4}{3}\pi R^3\right) = \frac{16}{3}\pi R^3$. So, the fraction of the box occupied by the balls is $\left(\frac{16}{3}\pi \cancel{R^3}\right)/(32\cancel{R^3}) = \frac{16\pi}{3\times 32} = \frac{\pi}{6} = \frac{\pi}{6} \times 100\,\% = 52.35\ldots\,\%$, which rounds to **52**%.

Team 5

The area enclosed by the circle being π means the radius of the circle is 1. Thus, the center of the circle is distance 1 from each straight segment of the quarter circle, so distance $\sqrt{2}$ from the corner where those two segments intersect. The distance from that intersection through the center of the circle to the quarter circle, which is the radius of curvature of the quarter circle, arc is $\sqrt{2} + 1$. Thus, the area enclosed by the quarter circle is $\pi(\sqrt{2} + 1)^2/4 = (3 + 2\sqrt{2})\pi/4$, and the shaded area is that minus π, or $(2\sqrt{2} - 1)\pi/4 = 1.436\ldots$, which rounds to the nearest 0.1 as **1.4**.

Team 6

Combining the two sets of inequalities yields $x/2 \le y \le \min(2x, 10)$, with $1 \le x \le 10$. For $x = 1$, y must be 1 or 2 [$\underline{2}$ pairs]. For $x = 2$, y must be 1, 2, 3, or 4 [$\underline{4}$ pairs]. For $x = 3$, y must be 2, 3, 4, 5, or 6 [$\underline{5}$ pairs]. For $x = 4$, y must be 2 through 8 [$\underline{7}$ pairs]. For x being 5 or 6, y must be 3 through 10 [$2 \times 8 = \underline{16}$ pairs]. For x being 7 or 8, y must be 4 through 10 [$2 \times 7 = \underline{14}$ pairs]. For x being 9 or 10, y must be 5 through 10 [$2 \times 6 = \underline{12}$ pairs]. Thus, we have a total of 2 + 4 + 5 + 7 + 16 + 14 + 12 = **60** ordered pairs.

Team 7

There are 6 points of crossing, and at each one, independently of the others, one ring can cross over or cross under the other ring participating in the crossing, thus 2 choices. For all 6 crossing points, there are $2^6 = 64$ total configurations. For two rings to be separable, one ring must be over the other at both points of crossing, and it could be either ring over the other, so 2 acceptable pairings. The third ring must cross over both other rings at all crossing point, or must similarly cross under both other rings, or can cross over one other ring at both points and cross under the remaining ring at both points, yielding 3 options. Thus, the overall probability is $\frac{2\times 3}{64} = \mathbf{\frac{3}{32}}$.

Team 8

Let's examine the complementary cases—at least 3 consecutive lights off. There are 7 lights, each with 2 states—on [+] or off [−]—independently of one another, totaling $2^7 = 128$ configurations. There are basically two such cases:

Case 1: 1 light on, followed by 3 lights off, followed by 3 lights each independently in either state [× for don't care]: +−−−×××; the 3 ×'s each independently contribute 2 options, and the on light that immediately precedes the at least 3 off lights can be any of the 7 lights [there cannot be two such lights, as each would be followed by at least three off lights, necessitating at least eight lights, but we have only seven, so no possibility of double counting], thus, $7 \times 1^1 \times 1^3 \times 2^3 = 56$ configurations.

Case 2: All lights off, which is 1 configuration.

Thus, the count of acceptable configurations is $128 - (56 + 1) = \mathbf{71}$.

Team 9

Let n notate a candidate 4-digit triskaidekaphilic number, and indicate its digits as KHTU, with $1 \le \text{K} \le 9$; indicate the digits of $n/13$ as htu. Being triskaidekaphilic requires KHTU to be 13 times one of the following: htu, ktu, khu, or kht. Removing one digit from such a number KHTU yields a 3-digit number unless $\text{K} = 1$ and is removed while $\text{H} = 0$; now we do have $n/13$ being a 2-digit number when $1001 \le n < 1300$, but we must have $\text{H} = 0$, so $1001 \le n < 1100$, in which case we have $n/13$ in 77 .. 84, so T is 7 or 8, but the only multiple of 13 satisfying those criteria is $1079 = 13 \times 83$, which fails being triskaidekaphilic. Thus, $1300 \le n \le 9999$ and $100 \le n/13 \le 769$. Multiplying htu by 13 will yield either $\text{K} > \text{h}$, or $\text{K} = \text{h}$ and $\text{H} > \text{t}$. In either case, at least one of the leftmost two digits of n will mismatch the two leftmost digits of $n/13$ and will have to be dropped. Because one, and only one digit is to be dropped from n, that means the rightmost two digits [TU] must match the 2 rightmost digits [tu] of $n/13$. Now, 3u ends in u if and only if u is 0 or 5. Likewise, u being 0 means that 13[t0] ends in [t0] if and only if t is 0 or 5. In a similar manner 13[t5] ends in [t5] if and only if t is 2 or 7. Thus, a triskaidekaphilic number must end in 00, 25, 50, or 75—in other words be a multiple of 25; by definition it must be a multiple of 13 as well—thus, a multiple of 325. That cuts the possibilities down to 27, namely $4 \times 325 = 1300$ through $30 \times 325 = 9750$. We can make a list of pairs starting with (100; 1300), successively adding (25; 325), and checking if one digit can be dropped from the right component to equal the left component—blue for triskaidekaphilic (with the dropped digit in **bold**), red for not:

(100; 1**3**00)	(200; 2**6**00)	(300; 3**9**00)	(400; 5200)	(500; 6**5**00)	(600; 7800)	(700; 9100)
(125; 1**6**25)	(225; 2**9**25)	(325; 4225)	(425; 5525)	(525; 6825)	(625; 8125)	(725; 9425)
(150; 1**9**50)	(250; 3**2**50)	(350; 4550)	(450; 5850)	(550; 7150)	(650; 8450)	(750; 9**7**50)
(175; 2275)	(275; 3575)	(375; 4875)	(475; 6175)	(575; 7475)	(675; 8775)	

The sum of the blue values is **33,800**.

Team 10

Let a, b, and c be the length of the sides of the right triangle opposite the vertices A, B, and C, respectively, as is customary; because A, not C, is the right angle, a, not c, is the hypotenuse. Choose a coordinate system with $\overleftrightarrow{\mathrm{AC}}$ and $\overleftrightarrow{\mathrm{AB}}$ being the x- and y-axes, respectively. Then A is at (0; 0), B is at (0; c), C is at (b; 0), D is $\frac{\sqrt{3}}{2}c$ to the left of the midpoint of $\overline{\mathrm{AB}}$ so $\left(-\frac{\sqrt{3}}{2}c;\ \frac{1}{2}c\right)$; E is $\frac{\sqrt{3}}{2}b$ below the midpoint of $\overline{\mathrm{AC}}$ so $\left(\frac{1}{2}b;\ -\frac{\sqrt{3}}{2}b\right)$; F is on the perpendicular bisector of $\overline{\mathrm{BC}}$ at a distance $\frac{\sqrt{3}}{2}a$ up and right along a slope $\frac{c}{b}$ from the midpoint $\left(\frac{1}{2}b;\ \frac{1}{2}c\right)$, thus $\left(\frac{1}{2}b + \frac{\sqrt{3}ac}{2\sqrt{b^2+c^2}};\ \frac{1}{2}c + \frac{\sqrt{3}ab}{2\sqrt{b^2+c^2}}\right) = \left(\frac{1}{2}b + \frac{\sqrt{3}}{2}c;\ \frac{1}{2}c + \frac{\sqrt{3}}{2}b\right)$ since $a = \sqrt{b^2 + c^2}$. Now having all the coordinates, we could use the surveyor's formula [often affectionately called the shoelace formula in MATHCOUNTS] to find the area of △DEF, but we can reduce the criss-cross effort substantially by translating the figure to move F to the origin by subtracting $\frac{1}{2}b + \frac{\sqrt{3}}{2}c$ from the x-components and $\frac{1}{2}c + \frac{\sqrt{3}}{2}b$ from the y-components of D, E, and F, resulting in $\mathrm{D}' = \left(-\frac{1}{2}b - \sqrt{3}c;\ -\frac{\sqrt{3}}{2}b\right)$, $\mathrm{E}' = \left(-\frac{\sqrt{3}}{2}c;\ -\frac{1}{2}c - \sqrt{3}b\right)$, $\mathrm{F}' = (0; 0)$. Thus, $\mathrm{Area}(\triangle\mathrm{DEF}) = \mathrm{Area}(\triangle\mathrm{D'E'F'}) = \frac{1}{2}\left|\mathrm{D}'_x\mathrm{E}'_y - \mathrm{D}'_y\mathrm{E}'_x\right| = \frac{\sqrt{3}}{4}b^2 + \frac{\sqrt{3}}{4}c^2 + \frac{5}{4}bc$. [As a strange sidenote, this area is the sum of the areas of the two smallest equilateral △, which is the same as the area of the largest equilateral △, plus 2.5 times the area of the right △, regardless of the size or shape of the starting right △—seems weird to me.] Now, we are given that this area is $23\sqrt{3}$ cm² and that $c = 2$ cm. Thus, $\frac{\sqrt{3}}{4}b^2 + \frac{5}{2}b + \sqrt{3} = 23\sqrt{3}$, so that $\frac{1}{2}b^2 + \frac{5}{\sqrt{3}}b - 44 = 0$. Thus, $b = -\frac{5}{\sqrt{3}} \pm \sqrt{\frac{25}{3} + 88} = -\frac{5}{\sqrt{3}} \pm \frac{17}{\sqrt{3}}$. The length b must be positive, so $b = \frac{12}{\sqrt{3}}$ cm $= 4\sqrt{3}$ cm. Finally, $a = \sqrt{b^2 + c^2} = \sqrt{48 + 4}$ cm $= \sqrt{52}$ cm $= \mathbf{2\sqrt{13}}$ cm.