MATHCOUNTS®

2023 Chapter Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide detailed solutions to the Sprint, Target and Team Rounds of the 2023 MATHCOUNTS Chapter Competition. These solutions show creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions with the MATHCOUNTS community.

Sprint 1
Peanut butter cost is $3 \times \$5 = \$15$. Jelly cost is $5 \times \$3 = \$15$. Total is $\$15 + \$15 = \$$**30** or $\$$**30.00**.

Sprint 2
$|2^5 - 5^2| = |32 - 25| = |7| = \mathbf{7}$.

Sprint 3
$\frac{1}{5} + \frac{11}{15} = \frac{3}{15} + \frac{11}{15} = \mathbf{\frac{14}{15}}$.

Sprint 4
$\$50 - \$10 - \$8 - \$7 = \$$**25** or $\$$**25.00**.

Sprint 5
Area of right triangle is $\frac{1}{2}$ of product of leg lengths: $\frac{1}{2} \times 8 \times 5 = \mathbf{20}$ cm^2.

Sprint 6
9 squares are shaded. Fraction shaded is $\frac{9}{25} = \frac{9\times4}{25\times4} = 36/100 = \mathbf{36}\%$.

Sprint 7
Starting point plus fraction of total distance: $0 + \frac{1}{3}(6 - 0) = \mathbf{2}$.

Sprint 8
$\frac{1}{2}\left(\frac{1}{3}\left(\frac{1}{4}(240)\right)\right) = \frac{1}{2}\left(\frac{1}{3}(60)\right) = \frac{1}{2}(20) = \mathbf{10}$.

Sprint 9
$y = -1$ when x is -3 or -1, of which the greater value is $\mathbf{-1}$.

Sprint 10
$4n + 1 = 250$, so $4n = 249$, so $n = \frac{249}{4} = 62\frac{1}{4}$, which rounds to **62**.

Sprint 11
$\sqrt{\frac{1}{4}} + \sqrt{\frac{1}{9}} + \sqrt{\frac{1}{16}} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6}{12} + \frac{4}{12} + \frac{3}{12} = \mathbf{\frac{13}{12}}$.

Sprint 12
Let l be the larger and s be the smaller. Then $l + s = 9$ and $l - s = 3$. Adding the two equations yields $2l = 12$, so $l = 6$. Substituting yields $6 + s = 9$, so $s = 3$. Therefore, the product is $ls = 6 \times 3 = \mathbf{18}$.

Sprint 13
Let n be the number of fish. We have $n = \frac{1}{2}n + \frac{1}{4}n + \frac{1}{8}n + 4 \rightarrow n = \frac{4}{8}n + \frac{2}{8}n + \frac{1}{8}n + 4 \rightarrow n = \frac{7}{8}n + 4 \rightarrow \frac{1}{8}n = 4 \rightarrow n = 8 \times 4 = \mathbf{32}$ fish.

Sprint 14
Let r be the expected number of red marbles. The fraction of marbles that are red is expected to be the same for the sample and the population, so $\frac{r}{100} = \frac{2}{10} = \frac{10\times2}{10\times10} = \frac{20}{100}$. Thus, $r = \mathbf{20}$ marbles.

Sprint 15
The number of used cups is $24(2 + 1) = 72$, and the number of unused cups is $100 - 72 = 28$. So, the number of tea parties she should host with three guests is $28/(3 + 1) = \mathbf{7}$ tea parties.

Sprint 16

The drive takes $\frac{75}{50} = 1.5$ hours $= 1$ hour 30 minutes. The 1 hour gets us from 12:11 a.m. to 1:11 a.m. The remaining 30 + 10 + 7 = 47 minutes gets us from 1:11 a.m. to **1:58** a.m or **01:58** a.m.

Sprint 17

The median of 5 ordered values is the third value. There are two values less than x, so we want the least of those greater than x, thus $x + 1$. The median is given as 6, so $x + 1 = 6$, and $x =$ **5**.

Sprint 18

Since the mean of x, y, and z is 99, we know that $x + y + z = 3 \times 99 = 297$. The value of z is maximized when the values of x and y are minimized. This occurs when x is the least positive integer, 1, and y is the next smallest integer, 2, so $z = 297 - 1 - 2 = \mathbf{294}$.

Sprint 19

An integer is divisible by 9 if and only if the sum of the digits in its base 10 representation is divisible by 9. Thus, 9 divides $1 + 2 + 3 + 4 + 5 + 6 + \text{E} = 21 + \text{E}$, so E must be **6**.

Sprint 20

Having 8 players playing for 1 hour = 60 minutes yields 8×60 player-minutes of activity. When those player-minutes are divided equally among 12 players, that works out to $\frac{8 \times 60}{12} = 8 \times 5 = \mathbf{40}$ minutes each.

Sprint 21

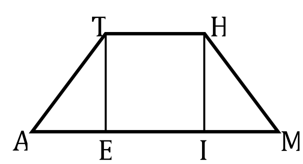
By the triangle inequality, $\text{AC} < 10 + 6 = 16$ and $\text{AC} > 10 - 6 = 4$. The **11** integer lengths between 4 and 16 are 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15. You can also think of this way: the triangle inequality theorem requires $|\text{AB} - \text{BC}| = 4 < \text{AC} < \text{AB} + \text{BC} = 16$, so there are $(16 - 1) - 4 = \mathbf{11}$ integers in that open interval.

Sprint 22

Each term in the sequence is twice the previous term. So, the first ten terms of this sequence are 1, 2, 4, 8, 16, 32, 64, 128, 256, 512. The average of these ten terms is $(1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512)/10 = 1023/10 = \mathbf{102.3}$. Alternatively, you may recall that the sum of the first n terms of a geometric sequence with first term a and common ratio r is $\frac{a_1 - a_{n+1}}{1-r}$. Here $a_k = 2^{k-1}$ and $r = 2$, so $a_1 = 2^0 = 1$ and $a_{11} = 2^{10} = 1024$ [many people having the latter fact memorized; if not, then $2^{10} = 2^{1+9} = 2^1 \times (2^3)^3 = 2 \times 8^3 = 2 \times 512$]. Thus, the sum is $\frac{1-1024}{1-2} = \frac{-1023}{-1} = 1023$, and the average is $\frac{1023}{10} = \mathbf{102.3}$.

Sprint 23

Drop auxiliary lines from T and H perpendicular to $\overline{\text{AM}}$, as shown, to form square THIE with TH = HI = EI = TE = 4 cm. Triangles TAE and HMI are congruent 3-4-5 right triangles with EA = MI = 3 cm. Thus, the perimeter of trapezoid MATH is $\text{MI} + \text{IE} + \text{EA} + \text{AT} + \text{TH} + \text{HM} = 3 + 4 + 3 + 5 + 4 + 5 = \mathbf{24}$ cm.



Sprint 24

You might happen to notice quickly that $8x + 10y = 3(x + 2y) + 1(5x + 4y) = 3(9) + 1(-4) = \mathbf{23}$, but if not, double the equation $x + 2y = 9$ and then subtract the equation $5x + 4y = -4$ to eliminate y and get $-3x = 22$, so $x = -22/3$ and $y = \frac{9-x}{2} = 49/6$. Thus, $8x + 10y = 8\left(-\frac{22}{3}\right) + 10\left(\frac{49}{6}\right) = \frac{-176+245}{3} = \frac{69}{3} = \mathbf{23}$.

Sprint 25

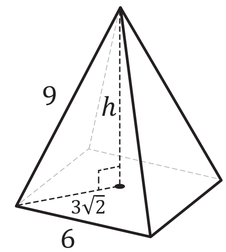
1 is 1 short of a package of 2; 2 is 1 short of a package of 3; 3 is 1 short of a package of 4. We need the integer that is 1 short of a multiple of the least common multiple of 2, 3, 4, that is 1 short of $12n$ for some integer n—we want the least such value exceeding 4 × 12 = 48, which occurs at $n = 5$, yielding 12 × 5 – 1 = **59** cupcakes.

Sprint 26

Use conjugation method. Now, $\sqrt{45} = 3\sqrt{5}, \sqrt{128} = 8\sqrt{2}, 2\sqrt{24} = 4\sqrt{6}$, so we have:

$\frac{3\sqrt{5}+2\sqrt{15}}{8\sqrt{2}+4\sqrt{6}} \times \frac{8\sqrt{2}-4\sqrt{6}}{8\sqrt{2}-4\sqrt{6}} = \frac{24\sqrt{10}-12\sqrt{30}+16\sqrt{30}-8\sqrt{90}}{128-96} = \frac{24\sqrt{10}+4\sqrt{30}-24\sqrt{10}}{32} = \frac{4\sqrt{30}}{32} = \boldsymbol{\frac{\sqrt{30}}{8}}$.

Sprint 27



Volume of pyramid is 1/3 of the product of the base area and the height. The base here is a square of perimeter 24 cm, so each edge of the square is 24 cm/4 = 6 cm and the base area is $6^2 = 36$ cm^2. The non-base edges have length $\frac{24-6}{2} = 9$ cm. As the figure shows, each lateral edge of the pyramid is the hypotenuse of a right triangle for which the center of the square base is the third vertex. By properties of 45-45-90 right triangles, we know that the square base has diagonal length $6\sqrt{2}$, so the shorter leg of this right triangle has length $\frac{6\sqrt{2}}{2} = 3\sqrt{2}$ cm. We can now use the Pythagorean theorem to determine the height h of the pyramid as follows: $\left(3\sqrt{2}\right)^2 + h^2 = 9^2 \rightarrow$

$h^2 = 9^2 - \left(3\sqrt{2}\right)^2 \rightarrow h^2 = 63 \rightarrow h = \sqrt{63} = 3\sqrt{7}$ cm. Now we can use the formula to calculat the volume of the pyramid as follows: $\frac{1}{3} \times 36 \times 3\sqrt{7} = \boldsymbol{36\sqrt{7}}$ cm^3.

Sprint 28

$x/|x|$ has value 1 when x is positive, −1 when x is negative, and undefined when $x = 0$. We have four cases for a, b, c.

Case 1: When all three are positive, the sum is $1 + 1 + 1 + 1 + 1 + 1 + 1 = \underline{7}$.

Case 2: When two are positive and one is negative, the sum is $-1 - 1 - 1 + 1 - 1 + 1 + 1 = -4 + 3 = \underline{-1}$.

Case 3: When one is positive and two are negative, the sum is $1 - 1 - 1 + 1 + 1 - 1 - 1 = -4 + 3 = \underline{-1}$.

Case 4: When all three are negative the sum is $-1 + 1 + 1 + 1 - 1 - 1 - 1 = -4 + 3 = \underline{-1}$.

The product of these outcomes is $7 \times (-1) \times (-1) \times (-1) = \boldsymbol{-7}$.

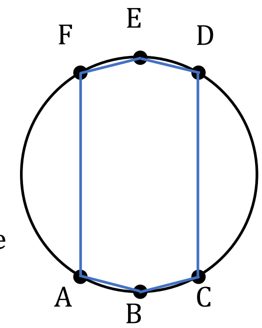
Sprint 29

Read carefully—only the perimeter, not the individual side lengths, needs to be an integer number of centimeters! The rectangle with the smallest perimeter whose area is 32 is a square, with side length $\sqrt{32} = 4\sqrt{2}$ cm and perimeter $4(4\sqrt{2}) = 16\sqrt{2}$ cm. This is not an integer, but $22 < 16\sqrt{2} < 23$. So, the smallest integer perimeter must be **23** cm. [These bounds are easy to determine if you know $\sqrt{2} \approx 1.4$ or you can determine them by squaring the $16\sqrt{2}$ term: $22^2 = 484 < \left(16\sqrt{2}\right)^2 = 512 < 23^2 = 529$] There was no need to determine the actual dimensions of the rectangle, but you can do so as follows: Let a and b be the dimensions of the rectangle. Then $ab = 32 \rightarrow b = \frac{32}{a}$ and $2(a + b) = 23$. Substituting for b in the second equation yields $2(a + \frac{32}{a}) = 23 \rightarrow a + \frac{32}{a} = \frac{23}{2} \rightarrow 2a^2 + 64 = 23a \rightarrow 2a^2 - 23a + 64 = 0$. Now, we can solve for a using the quadratic formula and see that

$a = 23 \pm \frac{\sqrt{(-23)^2-4(2)(64)}}{4} = 23 \pm \frac{\sqrt{529-512}}{4} = 23 \pm \frac{\sqrt{17}}{4}$. So, its dimensions are $\left(\frac{23+\sqrt{17}}{4}\right)$ cm $\times \left(\frac{23-\sqrt{17}}{4}\right)$ cm.

You can now confirm that the rectangle has area $\left(\frac{23+\sqrt{17}}{4}\right) \times \left(\frac{23-\sqrt{17}}{4}\right) = \frac{23^2-17}{16} = \frac{529-17}{16} = \frac{512}{16} = 32$ cm^2 and perimeter $2\left(\frac{23+\sqrt{17}}{4}\right) + 2\left(\frac{23-\sqrt{17}}{4}\right) = 2\left(\frac{46}{4}\right) = \frac{46}{2} = 23$ cm.

Sprint 30



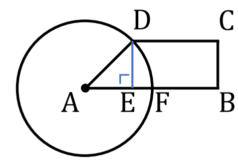
To qualify as a convex polygon, the sides must meet only at endpoints—no "crossing". If two nonconsecutive points are [directly] connected—say A and C, for example—then B must either connect to both A and C, or not. If B is connected to both A and C, then we have ABC forming a triangle with each vertex already connected to two other points, so there is no "room" for any of A, B, and C to connect to any of D, E, and F and two triangles, ABC and DEF, are formed, not a convex hexagon. If B is not connected to both of A and C, then B must have a connection to at least one of D, E, or F, and that line segment must cross $\overline{AB}$, violating the convexity requirement. Thus, each of the six points must be connected to the nearest point going each direction around the circle, yielding just one convex hexagon, ABCDEF. We can form a quasi-6-sided figure by starting at any one point, and iterating five times going to any remaining unvisited point, and finally going back to the first point: that means 6! options—but we have a factor 6 overcount due to being able to start at any of the six points and trace out the same path, and we have a factor 2 overcount due to being able to traverse the same path in either of two directions. Thus, we have $\frac{\cancel{6\times}5\times4\times3\cancel{\times2\times1}}{\cancel{6\times2}} = 60$ quasi-6-sided objects. We have also pairs of triangles, as described earlier with ABC and DEF, that meet the criterion about number of connections: A belongs to one of the triangles and it is connected to each of two other distinct points, meaning 5 choices for the first other point and 4 choices but we have a factor 2 overcount because the two other points could be chosen in either order; the three points other than A and the two points chosen to go with A automatically constitute the second triangle, so no new options for triangle pairs are created, and we have $\frac{5\times4}{2} = 10$ distinct triangle pairs. There cannot be a quasi-5-sided figure and an isolated point, because an isolated point has 0 connections, violating the requirement for 2; similarly, there cannot be a quasi-4-sided figure and a pair of points joined to each other. Altogether, there are $60 + 10 = 70$ figures. Thus, the probability of such a figure being a convex hexagon is $\mathbf{1/70}$.

Target 1
Let B be the price of a book and C be the price of crayons. We are given $15B + 3C = \$31.50$ and $B = C/2$. Thus, $C = 2B$ and $15B + 3(2B) = 21B = \$31.50$, so $B = \$31.50/21 = \$\mathbf{1.50}$.

Target 2
1 fortnight is 14 days, and 1 day $= 24\ \text{hours} \times \frac{60\ \text{min}}{1\ \text{hour}} \times \frac{60\ \text{sec}}{1\ \text{min}} = 86{,}400$ seconds, so one millionth of that is $14 \times 86{,}400/1{,}000{,}000 = 1.2096 \approx \mathbf{1.21}$ seconds.

Target 3
First $d + 8 = 15$, so $d = 7$. Then $c + d = 8$, so $c + 7 = 8$ and $c = 1$. Then $b + c = d$, so $b + 1 = 7$ and $b = 6$. Then $a + b = c$, so $a + 6 = 1$ and $a = -5$. Thus, $a + c = -5 + 1 = \mathbf{-4}$.

Target 4

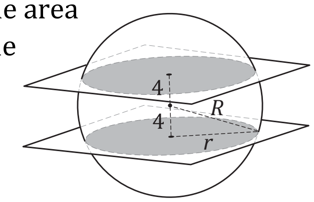


Drop an auxiliary line from D, perpendicular to $\overline{AB}$, intersecting $\overline{AB}$ at E. EB = DC = 9.6 cm, so AE = AB − EB = (14.9 − 9.6) = 5.3 cm = BC = ED. Thus, right triangle ADE is isosceles. So, $m\angle DAE = 45°$, meaning the sector is 45/360 = 1/8 of the circle. By properties of 45-45-90 right triangles, circle A has radius AD $= 5.3\sqrt{2}$ cm, so sector DAF has area $\frac{\pi}{8}\left(5.3\sqrt{2}\right)^2 \approx \mathbf{22.1}$ cm^2.

Target 5
Let the number of blocks from school to Wally's, Eli's, Yuko's and Sam's houses be W, E, Y and S, respectively. We have $E = \frac{1}{2}W$; $Y = W + E = \frac{3}{2}W$; $S = 3W$; $888 = W + E + Y + S = 6W$, so $W = \frac{888}{6} = \mathbf{148}$ blocks.

Target 6
There are 15 total letters: 5 As, 3 Ms, 2 Os, 3 Rs, 1 Z and 1 D. The probability of picking an A, followed by an M, followed by and O, followed by an R is $\frac{5}{15} \times \frac{3}{14} \times \frac{2}{13} \times \frac{3}{12} = \frac{1}{7\times13\times4}$. This probability is the same for any specific order they can be chosen in and there are $4 \times 3 \times 2 \times 1 = 24$ ways to order the four letter. So the probability of picking one A, one M, one O and one R, in any order, is $\frac{24}{7\times13\times4} = \frac{6}{7\times13} = \mathbf{\frac{6}{91}}$.

Target 7



Let r be the radius of each shaded circular region and R be the radius of the sphere. The area of each shaded region is $128\pi = \pi r^2$, so $r^2 = 128$ m^2. Because the two circles enclose the same area, they must be equidistant from the center of the sphere, thus 4 m from the center at closest approach. By Pythagoras, $R^2 = r^2 + 4^2 = 128 + 16 = 144$ m^2. The surface area of the sphere is $4\pi R^2 = \mathbf{576\pi}$ m^2.

Target 8
Let $\text{P}(k)$ be the probability of Kate giving a greeting on day k from 1 [Monday] to 5 [Friday]; $1 - \text{P}(k)$ is the probability of no greeting on day k. Then we have the recurrence relation $\text{P}(k+1) = \frac{3}{4}\text{P}(k) + \left(1 - \frac{3}{4}\right)\left(1 - \text{P}(k)\right) = \frac{1}{4} + \frac{1}{2}\text{P}(k)$. $\text{P}(1) = 1$, so $\text{P}(2) = \frac{1}{4} + \frac{1}{2}(1) = \frac{3}{4}$; $\text{P}(3) = \frac{1}{4} + \frac{1}{2}\left(\frac{3}{4}\right) = \frac{5}{8}$; $\text{P}(4) = \frac{1}{4} + \frac{1}{2}\left(\frac{5}{8}\right) = \frac{9}{16}$; $\text{P}(5) = \frac{1}{4} + \frac{1}{2}\left(\frac{9}{16}\right) = \mathbf{\frac{17}{32}}$. [One can prove a general formula of $\text{P}(k) = \frac{1}{2} + \left(\frac{1}{2}\right)^k$ for all positive integers k.]

Team 1
1 year = 365 days; 1 day = 24 hours; 1 hour = 4 kè. Thus, 1 year = 365 days = 365(24) hours = 365(24)(4) kè = **35,040** kè.

Team 2
Buses cannot be split into fractions, so our answer must be an integer. The total number of passengers is 270 + 24 = 294. Dividing by the number of passengers per bus yields 294/42 = 7, which is a whole number. Thus, they will need **7** buses.

Team 3
Powers of a prime do not work [p^n has proper factors p^k for $0 \le k \le n-1$ that sum to $\frac{p^n-1}{p-1} < p^n$] and the product of two distinct primes does not work [pq has proper factors 1, p, and q that sum to at most pq]. Thus, $25 = 5^2$, $26 = 2 \times 13$, and $27 = 3^3$ cannot work. Let's try $28 = 2^2 \times 7$: $(1+2+4)(1+7) - 28 = 28$ fails [perhaps you remember 28 is perfect, so not abundant]. 29 is prime, so let's try $30 = 2 \times 3 \times 5$: $(1+2)(1+3)(1+5) - 30 = 42 > 30$. So, the least abundant number greater than 24 is **30**.

Team 4
Let G, L, and R indicate a Green, bLue, and bRown eye, respectively. When a person has two colors, we do not care which eye has which color, so order is unimportant. Thus, we have 6 categories: GG, GL GR, LL, LR, RR. We are given GG + GL + GR = 400; GL + LL + LR = 600; GR + LR + RR = 900. Adding all these together yields (GG + LL + RR) + 2(GL + GR + LR) = 1900. We are given that the total population is 1200, so GG + LL + RR + GL + GR + LR = 1200, and 2(GG + LL + RR) + 2(GL + GR + LR) = 2400. Subtracting the equations 2(GG + LL + RR) + 2(GL + GR + LR) = 2400 and (GG + LL + RR) + 2(GL + GR + LR) = 1900 yields GG + LL + RR = 500. So, there are **500** residents.

Team 5
The mean of the 24 data items needs to be computed. That is quite a few values to sum. There are many techniques to make the task potentially a little easier, and different people have different preferences. The stem-and-leaf plot conveniently organizes the data in order by 10s and 1s. The 1s can be handled by adding all the 1s digits in the plot and skipping over the 0s: $8+7+2+2+4+4+5+6+6+8+8+8+5+5+8+9+2+2+3+3+5 = 110$. Then count how many are in each row to multiply the 10s value and add the products along with the 110 above: $110 + 1 \times 50 + 1 \times 60 + 11 \times 70 + 6 \times 80 + 5 \times 90 = 110 + 50 + 60 + 770 + 480 + 450 = 1920$. Divide that sum by the total count of 24 to get 1920/24 = 80 for the mean of the 24 scores. To keep the mean at 80 when adding 2 more scores means the new scores must have a mean of 80; to have a spread of 6 means one score is 3 (half of the 6 spread) above and the other 3 below the mean of 80. Here we want the greater score, so $80 + 3 =$ **83**.

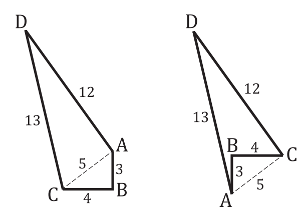
Team 6
Each cycle of slow-fast takes 1.5 minutes, so 55 minutes is $\frac{55}{3/2} = \frac{2}{3} \times 55 = 36\frac{2}{3}$ cycles, so 37 times for the 1 min slow portion and 36 times for the 0.5 minute fast portion. In one 1 min $= \frac{1}{60}$ hour slow portion, the speed is 5 mi/h , so $5 \times \frac{1}{60} = \frac{1}{12}$ mile is covered. Thus, over the 37 intervals of such accomplishment, $\frac{37}{12}$ mi is covered. The 36 portions at 0.5 minute each and $(5+x)\ \frac{\text{mi}}{\text{h}}$ totals $36 \times 0.5 \text{ min} = 18 \text{ min} = 0.3 \text{ h}$ in time and $0.3 \not{\text{h}} \times (5+x)\ \frac{\text{mi}}{\not{\text{h}}} = (1.5 + 0.3x)$ mi. Those two distance values combined yields a total of $\left(\frac{37}{12} + 1.5 + 0.3x\right)$ mi $= \left(\frac{37}{12} + \frac{18}{12} + 0.3x\right)$ mi $= \left(\frac{55}{12} + 0.3x\right)$ mi, which must equal the 5 mi total covered. Thus, $0.3x = 5 - \frac{55}{12} = \frac{5}{12}$, so $x = \frac{5/12}{3/10} = \frac{50}{36} = 1.3\overline{8}$, which rounds to **1.4**.

Team 7

Right triangles having integer lengths for all sides are Pythagorean triples (not necessarily primitive). A formula that generates all and only primitive Pythagorean triples involves two positive relatively prime integers m and n, with $m > n$ and one of which is even and the other odd. The shorter leg has length $m^2 - n^2$, the longer leg has length $2mn$, and the hypotenuse has length $m^2 + n^2$. Based on this, we get perimeter $P = m^2 - n^2 + 2mn + m^2 + n^2 = 2m^2 + 2mn = 2m(m + n)$. We are told that the perimeter must be less than 75, so $2m(m+n) \le 74 \rightarrow\ m(m+n) \le 37$. Starting with the least possible value for our smaller positive integer, we have $n = 1$ and $m(m+1) \le 37$, so $m \le 5$. Now that we know we can only consider values for m that are 5 or less, we can find all the possible ordered pairs (m, n) that can be used to generate the primitive Pythagorean triples for right triangles with perimeter less than 75. When $m = 5$, we have $5(5+n) \le 37 \rightarrow 25 + 5n \le 37 \rightarrow 5n \le 12 \rightarrow n \le 12/5$. Since n must be even, the only possible value is $n = 2$. When $m = 4$, we have $4(4+n) \le 37 \rightarrow 16\ +\ 4n \le 37 \rightarrow 4n \le 21 \rightarrow n \le 21/4$. Since n must be odd and $n < m$, we have $n = \{1, 3\}$. When $m = 3$, we have $3(3+n) \le 37 \rightarrow 9\ +\ 3n \le 37 \rightarrow 3n \le 28 \rightarrow n \le 28/3$. Since n must be even and $n < m$, we have $n = 2$. Finally, when $m = 2$, we have $2(2+n) \le 37 \rightarrow 4\ +\ 2n \le 37 \rightarrow 2n \le 33 \rightarrow n \le 33/2$. Since n must be odd and $m > n$, we have $n = 1$. So, we have the following ordered pairs which produce five primitive Pythagorean triples: (5, 2), (4, 1), (4, 3), (3, 2), (2, 1). We know that the triples produced by these ordered pairs all a sum less than 75, but multiples of these triples may also have a sum less than 75. The 3-4-5 triple, which yields a triangle with perimeter 12, produces a total of $\left\lfloor\frac{74}{12}\right\rfloor = \underline{6}$ right triangles (1 from the the original triple and 5 others by taking multiples of that triple). Similarly, the 5-12-13 triple produces a total of $\left\lfloor\frac{74}{30}\right\rfloor = \underline{2}$ right triangles; the 7-24-25 triple produces $\left\lfloor\frac{74}{56}\right\rfloor = \underline{1}$ right triangle; the 8-15-17 triple produces $\left\lfloor\frac{74}{40}\right\rfloor = \underline{1}$ right triangle; and the 20-21-29 triple produces $\left\lfloor\frac{74}{70}\right\rfloor = \underline{1}$ right triangle. That's a total of $6 + 2 + 1 + 1 + 1 = \mathbf{11}$ triangles.

Team 8

$\triangle$ACB is a 3-4-5 right triangle with area 6, and $\triangle$ACD is a 5-12-13 right triangle with area 30. The figures show the two quadrilaterals that meet the given specifications. The convex quadrilateral has area 30 + 6 = 36 units². And the concave triangle has area 30 − 6 = 24 units². We want the least possible area, which is **24** units².



convex quadrilateral   concave quadrilateral

Team 9

A total of 9 is achieved by, and only by, the following rolls: 1-2-6 with $\frac{3!}{1!\,1!\,1!} = \underline{6}$ orderings; 1-3-5 with $\frac{3!}{1!\,1!\,1!} = \underline{6}$ orderings; 1-4-4 with $\frac{3!}{1!\,2!} = \underline{3}$ orderings; 2-2-5 with $\frac{3!}{2!\,1!} = \underline{3}$ orderings; 2-3-4 with $\frac{3!}{1!\,1!\,1!} = \underline{6}$ orderings; 3-3-3 with $\frac{3!}{3!} = \underline{1}$ ordering. Thus, there are $6 + 6 + 6 + 3 + 3 + 1 = 25$ outcomes that meet the constraint of a total of 9, of which $6 + 6 + 6 = 18$ (from1-2-6, 1-3-5, 2-3-4) satisfy the additional criterion of all three dice having different values, for a probability of $\mathbf{\frac{18}{25}}$.

Team 10

1 is always a divisor, so we need exactly two other 1-digit divisors. Prime numbers do not qualify because they have only 2 positive divisors. The product pq of two distinct 1-digit primes works, as long as the product is at least 10 (so divisors 1, p, q), so 10, 14, 15, 21, and 35 work [count is 5 so far]. The power p^n of a prime works as long as $p^2 < 10$ but $p^3 \ge 10$ (so divisors 1, p, p^2), which works only for $p = 3$ and n is 3 or 4, thus 27 and 81 [count is 7 so far]. The product p^2q for primes p with $p^2 < 10$ and $q \ge 10$ (so divisors 1, p, p^2), which works for 44, 52, 68, 76, 92, and 99 [count is 13 so far]. The last category is the product p^2q for primes $p < 10$ and $q < 10$ but $p^2 \ge 10$ and $pq \ge 10$, which works for 50, 75, and 98 work [count is 16 so far. We are at the end of possibilities, so the answer is **16** integers.