MATHCOUNTS®

2022 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2022 MATHCOUNTS State Competition. These solutions provide creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions with the MATHCOUNTS community.

Sprint 1

Slope $= m = \frac{\Delta y}{\Delta x} = \frac{12-4}{5-3} = \frac{8}{2} = \mathbf{4}$.

Sprint 2

The high temperatures in degrees Fahrenheit on the five days were 10, 6, −3, −4, and 0. The mean high temperature, then, is $\frac{(10+6-3-4+0)}{5} = \frac{9}{5} = \mathbf{1.8}$ °F.

Sprint 3

We want an answer in square yards, but the data are given in feet. The number of feet for each dimension is easily divided by 3. We'll convert the data to yards and multiply, instead of multiplying feet with bigger numbers and then converting square feet to square yards. Since $33\text{ ft} = 33\,\cancel{\text{ft}} \times \frac{1\text{ yd}}{3\,\cancel{\text{ft}}} = 11\text{ yd}$ and $27\text{ ft} = 27\,\cancel{\text{ft}} \times \frac{1\text{ yd}}{3\,\cancel{\text{ft}}} = 9\text{ yd}$, the screen has area $11 \times 9 = \mathbf{99}\text{ yd}^2$.

Sprint 4

The terms of the sequence are as follows: $a_1 = 11$; $a_2 = 2$; $a_3 = 11 + 2 = 13$; $a_4 = 13 + 4 = 17$; $a_5 = 17 + 6 = 23$; $a_6 = 23 + 8 = 31$; $a_7 = 31 + 10 = 41$; $a_8 = 41 + 12 = 53$; $a_9 = 53 + 14 = 67$; $a_{10} = 67 + 16 = 83$; $a_{11} = 83 + 18 = 101$; $a_{12} = 101 + 20 = \mathbf{121} = 11^2$—finally a non-prime value.

Sprint 5

We need the least positive value of $12k$ that is a perfect cube, such that k is an integer. Because $12 = 2^2 \times 3^1$, k is of the form $2^m \times 3^n$ for some nonnegative integers m and n, so $12k = 2^2 \times 3^1 \times 2^m \times 3^n = 2^{m+2} \times 3^{n+1}$. To be a perfect cube, each exponent must be divisible by 3; the least such exponents are $m + 2 = 3, n + 1 = 3$. Therefore, we have $m = 1$ *and* $n = 2$. So, the desired value is $2^3 \times 3^3 = 8 \times 27 = \mathbf{216}$.

Sprint 6

The first palindrome is at 25952. Any 5-digit palindrome starting with 25 must end in 52 and we can have an arbitrary digit in the middle, but getting the next palindrome in the 25 thousands would require the next digit after 9 for the middle digit, but there is no such digit. Thus, we must go to the 26 thousands, and the palindrome must end in 62; the least middle digit is 0. The next palindrome after 25952 is 26062, with a separation of 110. Therefore, the travel is 110 miles in 2.5 hours, for an average speed of $\frac{110\text{ mi}}{2{,}5\text{ h}} = \mathbf{44}$ mi/h.

Sprint 7

Based on the rule given, we have $(1 \blacklozenge 2) \blacklozenge 3 = \frac{1\times2}{1+2} \blacklozenge 3 = \frac{2}{3} \blacklozenge 3 = \frac{\frac{2}{3}\times3}{\frac{2}{3}+3} = \frac{2}{11/3} = \mathbf{\frac{6}{11}}$.

Sprint 8

Let g be the initial weight of the grape, of which 80 %, for $0.8g$, is water, so $0.2g$ is not water. The grape loses half of its total weight, for $0.5g$, which is all water. Thus, the amount of water remaining in the grape is $0.8g - 0.5g = 0.3g$; the amount of non-water in the grape remains unchanged at $0.2g$. The portion of the dried grape that is water is $\frac{0.3g}{0.5g} \times 100\% = \mathbf{60}\%$.

Sprint 9

From the information given, we have $a = 5b$, so $15 = a + b = 5b + b = 6b$, making $b = \frac{15}{6} = \frac{5}{2}$. Therefore, $ab = 5b^2 = 5 \times \frac{25}{4} = \mathbf{\frac{125}{4}}$.

Sprint 10

The ordered pairs whose components are prime numbers with sum of 60 are: (7, 53), (13, 47), (17, 43), (19, 41), (23, 37), (29, 31). We see 6 ordered pairs, but we must also account for the reverse of each of these ordered pairs. Therefore, there are 6 × 2 = **12** ordered pairs.

Sprint 11

The slope of this function $y = f(x)$ is $\frac{f(10)-f(0)}{10-0} = 2(10+0) = 20$. Therefore, $f(10) - f(0) = 10 \times 20 = \mathbf{200}$.

Sprint 12

The surface area of the solid is the sum of the surface areas of the three cubes minus twice the overlap area among adjacent cubes. [The "twice" is due to one face on each of two adjacent cubes being obscured.] The surface area of a cube of edge length s is $6s^2$, so for the three cubes totals $6(1^2 + 2^2 + 3^2) = 6 \times 14 = 84\text{ cm}^2$. With each pair of adjacent cubes, the loss of surface area is twice the area of a face of the smaller cube, thus totaling $2(1^2 + 2^2) = 2 \times 5 = 10\text{ cm}^2$. The net surface area of the solid is, therefore, $84 - 10 = \mathbf{74}\text{ cm}^2$.

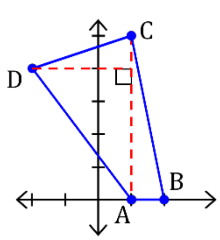
Sprint 13

When the operands of the two absolute values are either both nonnegative [occurring if and only if $x \geq 2$] or both negative [occurring if and only if $x < \frac{1}{2}$], the stated equation is equivalent to $2x - 1 = x - 2$, with unique solution $x = -1$ satisfying the constraint. When one operand is nonnegative and the other is negative [occurring if and only if $\frac{1}{2} \leq x < 2$], the stated equation is equivalent to $2x - 1 = -(x - 2)$, thus $3x = 3$ with unique solution $x = 1$ satisfying the constraint. Therefore, the sum of the solutions is $-1 + 1 = \mathbf{0}$.

Sprint 14

Let's express the weights as decimal numbers rather than percentages: $20\% = 0.20$ to be applied to the lowest pre-final exam score, which is 65%; $25\% = 0.25$ to be applied to the other pre-final exam scores, 80% and 92%. The sum of the weights must be $100\% = 1.00$, so the final exam must have a weight of $1.00 - (0.20 + 0.25 + 0.25) = 0.30$. We are to assume the maximum possible score of 100% for the final exam. The weighted average is $0.20 \times 65 + 0.25 \times 80 + 0.25 \times 92 + 0.30 \times 100 = 13 + 20 + 23 + 30 = \mathbf{86}\%$.

Sprint 15



The problem explicitly states that the polygon is convex, which is unusual. After plotting the four points, we see that the only one way to join these points with segments and create a convex polygon is in the following order A(1, 0), B(2, 0), C(1, 5), D(−2, 4). We observe that diagonal AC of convex quadrilateral ABCD is a vertical line segment, lying on $x = 1$. This diagonal splits ABCD into triangles ABC and ACD with a common base AC of length $5 - 0 = 5$. Triangle ABC with height $2 - 1 = 1$ has area $\frac{1}{2} \times 5 \times 1 = \frac{5}{2}$. Triangle ACD with height $1 - (-2) = 3$ has area $\frac{1}{2} \times 3 \times 5 = \frac{15}{2}$. Therefore, the area of ABCD is $\frac{5}{2} + \frac{15}{2} = \frac{20}{2} = \mathbf{10}$ units^2.

Sprint 16
We are told that $xy^2 = 6$ and $x^2y^6 = 72$. Cubing the first equation gives us $x^3y^6 = 216$. Dividing this new equation by $x^2y^6 = 72$, we get $\frac{x^3y^6}{x^2y^6} = \frac{216}{72}$. $So, x = 3$. Now, substituting into the original equation, we get $3y^2 = 6$. So, $y^2 = 2$ and $y = \sqrt{2}$. Therefore, $xy = \mathbf{3\sqrt{2}}$.

Sprint 17
Squaring both sides of $\sqrt{7 - \sqrt{2 + \sqrt{n}}} = 2$, we get $7 - \sqrt{2 + \sqrt{n}} = 4$, and $\sqrt{2 + \sqrt{n}} = 3$. Squaring both sides yields $2 + \sqrt{n} = 9$, so $\sqrt{n} = 7$. Square both sides one last time to obtain $n = 49$. Because we have squared both sides three times, we have had plenty of opportunities to generate extraneous roots, so it would be a good idea to check that 49 really works: $\sqrt{7 - \sqrt{2 + \sqrt{49}}} = \sqrt{7 - \sqrt{2 + 7}} = \sqrt{7 - \sqrt{9}} = \sqrt{7 - 3} = \sqrt{4} = 2$, so, indeed **49** works.

Sprint 18
The numbers 2, 5, and 7 are pairwise relatively prime and are divisors of 280, making the conditions independent of one another. Thus, we can say $\frac{1}{2}$ of the values in 1 … 280 are divisible by 2, so the other $\frac{1}{2}$ are not; $\frac{1}{5}$ are divisible by 5, so the other $\frac{4}{5}$ are not; $\frac{1}{7}$ are divisible by 7, so the other $\frac{6}{7}$ are not. Because of independence, we can multiply the three "are not" quantities to see that we are talking about $\frac{1}{2} \times \frac{4}{5} \times \frac{6}{7} = \frac{12}{35}$ of the 280 values. So, $\frac{12}{35} \times 280 = 12 \times 8 = \mathbf{96}$ integers in that range are not divisible by any of 2, 5, or 7.

Sprint 19
The area enclosed by a semicircle of radius r is given by $\pi r^2/2$. Here the diameter is given as 20, so the radius is half that, or 10, and the enclosed area is $100\pi/2 = 50\pi$. One side of the triangle is a diameter of the semicircle, so it is a right triangle. With one leg being 12 and hypotenuse being 20, we must have a 3-4-5 triangle with scale factor 4. So, the second leg has length 16. The area enclosed by a right triangle is half the product of the legs, which here is $\frac{1}{2} \times 12 \times 16 = 96$. So, the shaded area is $50\pi - 96$. Therefore, $a = 50$ and $b = 96$, and $a + b = 50 + 96 = \mathbf{146}$.

Sprint 20
We have $n = (10^{2020} - 10^{2019}) + (10^{2018} - 10^{2017}) + \cdots + (10^2 - 10^1) = 9 \times 10^{2019} + 9 \times 10^{2017} + \cdots + 9 \times 10^1$, which shows that we have a 2020 digit number whose digit in the 10^n place is 9 when n is odd (which is 1010 times) and 0 when n is even (which is 1010 times). Therefore, the sum of the digits is $1010 \times 9 + 1010 \times 0 = \mathbf{9090}$.

Sprint 21
The only way to falsify this claim by the renowned Welsh mathematician-philosopher Bertrand Russell is to find one example of a violation of the conditional statement. The only occasion for which the conditional statement "If p, then q" [in symbols $p \Rightarrow q$] is violated is when p is true but q is false. Thus, our founder of set theory, German mathematician Georg Cantor, needs to turn over every card showing an even number to make sure a circle is on the other side (if it is not for any card, then Bertrand lies) as well as every card showing a shape other than circle to make sure an odd number is on the other side (if it is not for any card, then Bertrand lies). There are 4 cards with an even number showing and 3 cards with a non-circular shape, thus, $4 + 3 = \mathbf{7}$ cards to check.

Sprint 22

There is only one right triangle with hypotenuse 10 cm and a leg of 6 cm or 8 cm, namely a 3-4-5 triangle with scale factor 2 cm. Thus, the four triangular bases among the two prisms are congruent, so all variation in the volume of the two prisms is due solely to differences in height. That means $\frac{x-2}{x} = 75\ \% = \frac{3}{4}$, so $3x = 4x - 8$, so $x = \mathbf{8}$ cm.

Sprint 23

$11! = 11 \times (2 \times 5) \times 3^2 \times 2^3 \times 7 \times (2 \times 3) \times 5 \times 2^2 \times 3 \times 2 = 2^8 \times 3^4 \times 5^2 \times 7^1 \times 11^1$, which has $(8+1)(4+1)(2+1)(1+1)(1+1) = 9 \times 5 \times 3 \times 2 \times 2 = 27 \times 20 = 540$ divisors. The only prime divisors are 2, 3, 5, 7, and 11—thus, 5 of them. Therefore, the probability of a randomly chosen divisor of 11! being prime is $\frac{5}{540} = \mathbf{\frac{1}{108}}$.

Sprint 24

Thomas can choose any 4 out of 12 players, which is ${}_{12}C_4$ (read "12 choose 4") distinct possibilities for a team. Carrie can choose any 4 out of the remaining 8 players, which is ${}_{8}C_4$ distinct possibilities for a team. Lenny has only 1 choice for his team, whichever 4 have not yet been chosen. Combining all this yields $\frac{\cancel{12\times}11\times10\times9}{\cancel{4\times3\times2}} \times \frac{\cancel{8\times}7\times\cancel{6\times}5}{\cancel{4\times3\times2}} = 99 \times 35 \times 10 = (3500 - 35) \times 10 = \mathbf{34{,}650}$ ways.

Sprint 25

Since $45 = 3^2 \times 5$, we need to find how many factors of 3 are in 80! and divide by 2 to count the factors of 9, along with how many factors of 5 are in 80! The lesser of the two counts is the desired result. Count of 3s as factors: $\left\lfloor\frac{80}{3}\right\rfloor = 26$; $\left\lfloor\frac{26}{3}\right\rfloor = 8$; $\left\lfloor\frac{8}{3}\right\rfloor = 2$, so $26 + 8 + 2 = 36$ factors of 3, thus 18 factors of 9. Count of 5s as factors: $\left\lfloor\frac{80}{5}\right\rfloor = 16$; $\left\lfloor\frac{16}{5}\right\rfloor = 3$, so $16 + 3 = 19$ factors of 5. [The notation $\lfloor x \rfloor$ means the floor of the real number x, which is the greatest integer less than or equal to x, so $\left\lfloor\frac{x}{n}\right\rfloor$ gives how many multiples of n are in the range $1 \ldots x$.] The lesser value of 18 and 19 is **18**.

Sprint 26

Simplifying the expression on the left-hand side of the equation yields $(7^2 + 24^2)^4 \times (5^2 + 10^2)^5 \times (75^2 + 100^2)^6 = (25^2)^4 \times 125^5 \times (125^2)^6 = 5^{2\times2\times4} \times 5^{3\times5} \times 5^{6\times6} = 5^{16+15+36} = 5^{67}$. So, $n = \mathbf{67}$.

Sprint 27

Factoring the quadratic expression, we get $m^2 + 14m - 32 = (m + 16)(m - 2) = 3^n$. So, each of the two factors must be a power of 3. The only powers of 3 that differ by 18 are $9 = 3^2$ and $27 = 3^3$. Therefore, $m = 11$, $n = 2 + 3 = 5$, and $m + n = 11 + 5 = \mathbf{16}$.

Sprint 28

The sum of the roots of the polynomial $p(x) = ax^2 + bx + c$ must be $-b/a$. We can easily add $-0.19098301 - 1.30901699 = -1.5$, so $\frac{b}{a} = -(-1.5) = 1.5 = \frac{3}{2}$. Thus, $2b = 3a$, so a must be even and b must be divisible by 3. This means b must be 3, 6, or 9, so a must be 2, 4, or 6, respectively. The product of the roots of $p(x)$ must be c/a. This one is a bit harder without a calculator. Let's turn up the brain power, as we are not going to multiply those messy values with pencil and paper. The negative signs will cancel each other in the product, so we can ignore the signs in analyzing the product. Now, $0.19 < 0.1909\ldots < 0.2$ and $1.3 < 1.3090\ldots < 1.31$, so the product must satisfy $0.247 = 0.19 \times 1.3 < \frac{c}{a} < 0.2 \times 1.31 = 0.262$. The only fraction in that range whose numerator and denominator are 1-digit integers is $\frac{1}{4}$. [The next closest values are $\frac{2}{9}$ on the low side and $\frac{2}{7}$ on the high side, both out of bounds.] Thus, $a = 4c$, so a must be 4 or 8. Only $a = 4$ satisfies both the b and the c requirements. Therefore, $a = 4$, $b = 6$, $c = 1$, and $p(x) = 4x^2 + 6x + 1$. Thus, $p(10) = 4 \times 10^2 + 6 \times 10 + 1 = \mathbf{461}$.

Sprint 29

The expression $\frac{11^{20} + 9^{20}}{11^{20} - 9^{20}}$ can be rewritten as $1 + \frac{2 \times 9^{20}}{11^{20} - 9^{20}} = 1 + \frac{2}{(11/9)^{20} - 1^{20}}$. We want the hundredths digit of this expression. The "1" term does not affect the fractional part, so we'll disregard it. To find the hundredths digit of x, find the ones digit of $100x$. Thus, we want the ones digit of $\frac{200}{(11/9)^{20} - 1}$ (not rounded). We estimate $\left(\frac{11}{9}\right)^{20} = \left(\left(\frac{11}{9}\right)^2\right)^{10} = \left(\frac{121}{81}\right)^{10} \approx \left(\frac{3}{2}\right)^{10}$. And $\left(\frac{3}{2}\right)^{10} = \left(\left(\frac{3}{2}\right)^4\right)^2 \left(\frac{3}{2}\right)^2 = \left(\frac{81}{16}\right)^2 \times \frac{9}{4} \approx 5^2 \times \frac{9}{4}$. Finally, $5^2 \times \frac{9}{4} = \frac{225}{4} = 56.25$. (The actual value is 55.335.) Now, divide by hand $\frac{200}{56.25 - 1} = \frac{800}{221} = 3.6\ldots$. Therefore, the desired digit is **3**. (Note: The 56.25 that is about 1.6 % too high means that the result of dividing by approximately that value is about 1.6 % too low. An approximation error of less than 2 % too low, means the true answer is less than 0.074 higher, meaning the ones digit must, indeed, be 3, with room to spare.)

Sprint 30

Since we are looking for an integer value, each of the prime numbers 2, 3, and 5 occur as factors an even number of times (8, 4, and 2, respectively), so it is conceivable for each of 2, 3, and 5 to split with half of the factors in the numerator canceling half in the denominator. The prime number 7 occurs only once in the expression, so 7 must end up in the numerator of the overall simple fraction. It looks like the best we can possibly do is 7. So, if we can find an arrangement of parentheses yielding a value of 7, then we are done. In terms of a simple common fraction, 1 must go on top and 2 on the bottom regardless of parentheses; 7 must be on top for an integer result. Try 8 on the bottom with the 2; all others with factors of 2 (4, 6, 10) must go on top; 6 on top forces 3 on top and 9 on bottom; 10 on top forces 5 on the bottom. So, we have the expression $\left(\left(\left(1 \div ((2 \div 3) \div 4)\right) \div ((5 \div 6) \div 7)\right) \div 8\right) \div (9 \div 10)$, *which equals* **7**.

Target 1
The first card with F is #4 (DEFG). Then F appears on every card until we pass F and start with G, thus cards starting with D, E, or F for **3** cards.

Target 2
The area of quadrilateral ECBF is the area of square ABCD minus the combined areas of triangles ADE and BFA. Square ABCD has side length 2 cm and area $2^2 = 4$ cm^2. Triangle ADE is a right triangle with legs of lengths AD = 2 cm and DE = $\frac{1}{2} \times$ DC = $\frac{1}{2} \times 2 = 1$ cm and area $\frac{1}{2} \times 2 \times 1 = 1$ cm^2. Angle DEA and angle BAE are alternate interior angles and, therefore, equal in measure. Furthermore, since AB⊥AD and BF⊥AE, it follows that triangles BFA and ADE are similar. Since AE = $\sqrt{(1^2 + 2^2)} = \sqrt{5}$, the ratio of corresponding sides of triangles BFA and ADE is 2 to $\sqrt{5}$, meaning the ratio of their areas is 4 to 5. The enclosed area of triangle BFA, then, is $\frac{4}{5}$ the area of triangle ADE, or $\frac{4}{5}(1) = 0.8$ cm^2. Therefore, the area of ECBF is $4 - 1 - 0.8 = \mathbf{2.2}$ cm^2.

Target 3
Because the total percentage of families is 100%, the childless families must account for $x = 100 - 20 - 18 - 10 - 6 = 46\% = 0.46$. The number of families with no children, then, is $0.46 \times 10{,}250 = \mathbf{4715}$ families.

Target 4
The curves $y = x^2 - x$ and $y = 3x + a$ intersect where the two are equal, $x^2 + x = 3x + a$, thus when $x^2 - 2x - a = 0$. The quadratic formula indicates solutions $x = 1 \pm \sqrt{1 + a}$. The horizontal separation of the two points of intersection with coordinates $(1 - \sqrt{1+a}, y_1)$ and $(1 + \sqrt{1+a}, y_2)$ is the positive difference of these two solutions, thus $(1 + \sqrt{1+a}) - (1 - \sqrt{1+a}) = 2\sqrt{1+a}$. Because the slope of the line is 3, the vertical separation of the two intersections is 3 times the horizontal separation. So, $y_2 - y_1 = 6\sqrt{1+a}$. By the Pythagorean theorem, the distance between the two points of intersection is $\sqrt{\left(2\sqrt{1+a}\right)^2 + \left(6\sqrt{1+a}\right)^2}$ $= \sqrt{4 + 4a + 36 + 36a} = \sqrt{40 + 40a} = 2\sqrt{10(1+a)}$, which we are told is $3\sqrt{30}$. So, $2\sqrt{10(1+a)} = 3\sqrt{30}$. Solving for a, we have $40(1 + a) = 270$, so $1 + a = \frac{27}{4}$, and $a = \frac{23}{4}$. Because of our squaring, we should check that our solution is not extraneous: $2\sqrt{10\left(1 + \frac{23}{4}\right)} = 2\sqrt{10\left(\frac{27}{4}\right)} = \sqrt{270} = 3\sqrt{30}$, so the solution is $\mathbf{\frac{23}{4}}$.

Target 5
We can multiply to get $200\ \cancel{\text{shp}} \times \frac{9.9\ \cancel{\text{lb}}}{1\ \cancel{\text{shp}}} \times \frac{10.5\ \cancel{\text{mi}}}{1\ \cancel{\text{lb}}} \times \frac{1760\ \cancel{\text{yd}}}{1\ \cancel{\text{mi}}} \times \frac{1\ \text{skn}}{175\ \cancel{\text{yd}}} = 218048.91\ldots$ skn, which rounded to the nearest 1000 skeins is **218,000** skeins.

Target 6
Let's label the sets $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ as our universe, $E = \{2, 4, 6, 8, 10, 12\}$, $T = \{3, 6, 9, 12\}$, $P = \{2, 3, 5, 7, 11\}$, and S as any arbitrary subset of U under consideration. For each of the sets E, T, and P, at least one of 2 and 3 is an element, so let's split S into 4 cases:
(1) $2 \in S, 3 \in S$: There is a non-empty overlap with each of the three sets, regardless of whether the other 10 elements of U are in or out of S, so $2^{10} = \underline{1024}$ options.
(2) $2 \in S, 3 \notin S$: We need at least one of 6, 9, and 12 in S to have non-empty overlap with T; each of the remaining 7 elements of U can be in or out of S, so $(2^3 - 1)2^7 = \underline{896}$ options.
(3) $2 \notin S, 3 \in S$: We need at least one of 4, 6, 8, 10, and 12 in S to have non-empty overlap with E; each of the remaining 5 elements of U can be in or out of S, so $(2^5 - 1)2^5 = \underline{992}$ options.
(4) $2 \notin S, 3 \notin S$: We need at least one of 5, 7, and 11 to have non-empty overlap with P; we need at least one

of 6, 9, and 12 to have non-empty overlap with T, but there is an issue of dependence with E because 6 and 12 are in E but 9 is not, so we need to split into 2 subcases:
(4a) $6 \in S$ or $12 \in S$: This takes care also of non-empty overlap with E, so the remaining 5 elements of U (1, 4, 8, 9, 10) can be in or out of S, so $(2^3 - 1)(2^2 - 1)2^5 = \underline{672}$ options.
(4b) $6 \notin S$ and $12 \notin S$: Then $9 \in S$, and at least one of 2, 4, and 8 must be in S to have non-empty overlap with E. The element 1 can be in or out of S, so $(2^3 - 1)(2^3 - 1)2^1 = \underline{98}$ options.
Adding up all the options yields $1024 + 896 + 992 + 672 + 98 = \mathbf{3682}$ subsets.

Target 7

Ashley and Bernard mismatched answers on #1 and #3, so one got #1 correct and the other was incorrect, and similarly for #3. Because Ashley and Bernard each missed one, one of them must have missed #1 and the other missed #3, both with correct answers on all the other questions. We see that Clive matched their answers on #5 and #6, but not on #2, #4, #7, and #8, so Clive got 2 of those 6 correct. Clive matched Bernard on #1 and #3, one of which was correct, so Clive likewise got 1 of those 2 correct, for a total of **3** questions correct.

Target 8

The constant term of a quadratic equation equals the product of the two roots and the leading [quadratic] coefficient. Let the roots of the quadratic equation $x^2 + ax + b = 0$ be r_1 and r_2, which we are told are distinct integers. Thus, $p(x) = x^2 + ax + b = (x - r_1)(x - r_2)$, so that $b = r_1 r_2$. Because all three coefficients of the equation are stated to be positive, that means both roots are negative. Thus, our factored equation involves subtracting negative numbers, which can get somewhat awkward, so define $R_1 = -r_1$ and $R_2 = -r_2$, with consequence that $b = r_1 r_2 = (-R_1)(-R_2) = R_1 R_2$, which is what we are to minimize. We are given also that $p(60) = (60 + R_1)(60 + R_2)$ is a perfect square. Since R_1 and R_2 are distinct positive integers, we can choose without loss of generality that $1 \le R_1 < R_2$.
$R_1 = 1$; $61(60 + R_2)$ is perfect square, so $R_2 \ge 4 \times 61 - 60 = 184$ and $R_1 R_2 = 184$. Best so far.
$R_1 = 2$; $62(60 + R_2)$ is perfect square, so $R_2 \ge 4 \times 62 - 60 = 188$ and $R_1 R_2$ is worse.
$R_1 = 3$; $63(60 + R_2)$ is perfect square, so $R_2 \ge 16 \times 7 - 60 = 52$ and $R_1 R_2 = 156$. Best so far.
$R_1 = 4$; $64(60 + R_2)$ is perfect square, so $R_2 \ge 1 \times 81 - 60 = 21$ and $R_1 R_2 = 84$. Best so far.
$R_1 = 5$; $65(60 + R_2)$ is perfect square, so $R_2 \ge 4 \times 65 - 60 = 200$ and $R_1 R_2$ is worse.
$R_1 = 6$; $66(60 + R_2)$ is perfect square, so $R_2 \ge 4 \times 66 - 60 = 204$ and $R_1 R_2$ is worse.
$R_1 = 7$; $67(60 + R_2)$ is perfect square, so $R_2 \ge 4 \times 67 - 60 = 208$ and $R_1 R_2$ is worse.
$R_1 = 8$; $68(60 + R_2)$ is perfect square, so $R_2 \ge 9 \times 17 - 60 = 93$ and $R_1 R_2$ is worse.
$R_1 \ge 9$ means $R_2 \ge 10$, so $R_1 R_2 \ge 90$ is worse.
Therefore, the least possible value for b is $\mathbf{84}$.

Team 1
We can convert the times from the 12-hour clock to the 24-hour clock for easier computation:
9:33 a.m.→09:33; 5:20 p.m.→17:20; 12:18 p.m.→12:18; 12:41 p.m.→12:41. The break lasted 12:41 – 12.18 = 0:23, thus 23 minutes. If we delay the start time from 09:33 by 0:23 to 09:56 and take away the 23-minute break, we have the same total working time. Now, 09:56 is 4 minutes before 10:00, which in turn is 7:20 before 17:20. Therefore, the total working time is 4 mintes more than 7 hours 20 minutes, for a total of $7\frac{24}{60} =$ 7.4 hours, so the total pay was $7.4\ \cancel{h} \times \frac{\$15}{\cancel{h}} = \$\mathbf{111}$ **or** $\$\mathbf{111.00}$.

Team 2
The most straightforward way to end the game as quickly as possible is to have tokens completely taken away from players, which is achieved with a roll of 1 and a token going to the middle, not to another player. If the roll is 1 on each of the first 6 turns, then each player is down to 1 token. Then, if the seventh roll is 1, the first player has none left, while the other two players still have 1 token left. Then, if the eighth roll is 1, the first two players have none left, and only the third player has any remaining tokens, thus ending the game. Therefore, the minimum number of turns is **8** turns.

Team 3
There are 2 games with 3 goals. There are 8 games with at least 2 goals, so $8 - 2 = 6$ games with exactly 2 goals. There are 24 games with at least 1 goal, so $24 - 8 = 16$ games with exactly 1 goal. There are $50 - 24 = 26$ games with 0 goals. Therefore, the average is $\frac{2\times3 + 6\times2 + 16\times1 + 26\times0}{50} = \frac{6 + 12 + 16 + 0}{50} = \frac{34}{50} = \mathbf{0.68}$ goals per game.

Team 4
The least common multiple of 6 and 4 is 12, so the two counts overlap every multiple of 12. Between 2000 and 3000 inclusive, the first number they both say aloud is 2004 and the last is 3000. Between 2004 and 3000 there are (3000 − 2004)/12 = 996/12 = 83 multiples of 12. But this does not count 3000. So, there are 83 + 1 = **84** numbers said by them both.

Team 5
A regular octahedron has 6 vertices, each of which is the intersection of 4 edges. For the ending vertex being the same as the starting vertex, the distance is 0, which is not greater than 1. There are 4 adjacent vertices for a distance of 1, which, again is not greater than 1. There is 1 vertex opposite the starting vertex for a distance $\sqrt{2}$, which is greater than 1. Thus, we need the probability of being at the opposite vertex at the end of three moves. The only type of path that will be on the opposite vertex at the end of three moves has a first move to any of the 4 vertices adjacent to A (probability 1), a second move to either of the 2 adjacent vertices that also are adjacent to A out of the 3 adjacent vertices (probability 2/3), a third move to the bottom vertex, which is 1 of 3 possible vertices to move to (probability 1/3). The overall probability is the product of these three single-move probabilities: $1 \times \frac{2}{3} \times \frac{1}{3} = \mathbf{\frac{2}{9}}$.

Team 6

The powers of 3 less than 29 are 1, 3, 9 and 27. As the table below shows, there is 1 3-partion using 27. For that 3-partition, replacing the one occurrence of 27 with three 9s yields 1 3-partition. For that last 3-partition, replacing one or more of the 3 occurrences of 9 with three 3s yields 3 3-partitions. For each of those last three 3-partitions, replacing one or more of the 18 occurrences of 3 with three 1s yields 18 3-partitions.

27s	1		0		0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9s	0		3		2	1	0		2	2	2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
3s	0		0		3	6	9		2	1	0	5	4	3	2	1	0	8	7	6	5	4	3	2	1	0
1s	2		2		2	2	2		5	8	11	5	8	11	14	17	20	5	8	11	14	17	20	23	26	29

That's a total of 1 + 1 + 3 + 18 = **23** 3-partitions.

Team 7

Group n has $n+2$ terms. The last term of the nth group has the value $2 \times \big(3+4+\cdots+(n+2)\big) = 2 \times \frac{(3+(n+2))n}{2} = n(n+5)$; the minimum term in the nth group is $2n+2$ less, and the mean of all terms in the nth group is $n+1$ less for n^2+4n-1, so with $n+2$ terms, the sum is $(n^2+4n-1)(n+2)$. For $n = 100$, this sum is $10{,}399 \times 102 = \mathbf{1,060,698}$.

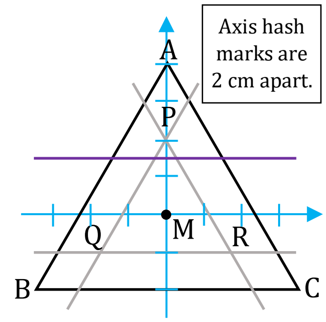
Team 8

Using a single color, there are 4 ways to paint the tetrahedron. There are two cases to consider if two colors are used to paint the tetrahedron: three faces one color and one face another, two faces one color and two faces another. For the first case, there are 4 choices of color for the three faces and 3 choices for the fourth face, giving us $4 \times 3 = 12$ ways to paint it. For the second case, there are 4 choices of color for the first two faces and 3 choices for the remaining two faces. Accounting for duplicate paintings there are $(4 \times 3)/2 = 6$ ways to paint the tetrahedron in this manner. Using three colors, with two faces painted one color and each of the remaining faces painted two different colors, there are four choices for the color of the first two faces, three choices for the color of the third face and two choices for the color of the fourth face. Accounting for duplicate paintings, there are $(4 \times 3 \times 2)/2 = 12$ ways to paint the tetrahedron. Finally, if all four colors are used, one color per side there are 2 ways to color the tetrahedron. If the red side is placed down on the table, then looking down from above will show blue-green-yellow or yellow-green-blue in clockwise order Therefore, there are $4 + 12 + 6 + 12 + 2 = \mathbf{36}$ distinct paintings of the tetrahedron.

Team 9

We'll refer to a sequence of n consecutive coin tosses with the same outcome (heads or tails) as a run of n. We want to determine the probability of 10 consecutive tosses that contain only runs of 1 or 2. There are 6 cases based on the number of runs of 2 in the 10 tosses, which can be any count t satisfying $0 \le t \le 5$. The counterpart count u of runs of 1 is $u = 10 - 2t$. The number of distinct arrangements of t runs of 2 and u runs of 1 is ${}_{t+u}C_t$. For $t = 0, u = 10, \frac{(0+10)!}{0!\,10!} = 1$. For $t = 1, u = 8, \frac{(1+8)!}{1!\,8!} = 9$. For $t = 2, u = 6, \frac{(2+6)!}{2!\,6!} = 28$. For $t = 3, u = 4, \frac{(3+4)!}{3!\,4!} = 35$. For $t = 4, u = 2, \frac{(4+2)!}{4!\,2!} = 15$. For $t = 5, u = 0, \frac{(5+0)!}{5!\,0!} = 1$. The total number of acceptable arrangements of runs is $1 + 9 + 28 + 35 + 15 + 1 = 89$. Now, each sequence of coin tosses may start with either head or tail, thus 2 options applying to each arrangement of runs. Thus, there are $2 \times 89 = 178$ acceptable sequences of 10 coin tosses. There are $2^{10} = 1024$ possible sequences of 10 coin tosses. Thus, the probability of 10 coin tosses with no runs more than 2 is $\frac{178}{1024} = \mathbf{\frac{89}{512}}$.

Team 10



The Viviani theorem says that the sum of the distances from any point interior to an equilateral triangle to the three sides of that triangle equals the altitude of that triangle, 12 cm for us. Thus, special points must be 7 cm from one side, 3 cm from a second side, and 2 cm from the third side; if any two are true, the third must also be true. Draw $\triangle$ABC on an xy-plane with centroid M at $(0, 0)$ and the bottom side parallel to the x-axis. In terms of number of centimeters, A is at $(0, 8)$, B at $\left(-4\sqrt{3}, -4\right)$, C at $\left(4\sqrt{3}, -4\right)$. Each of the three light gray lines forming $\triangle$PQR is 2 cm interior to the corresponding parallel side of $\triangle$ABC, so the special points must be on $\triangle$PQR. Use 3-fold rotation symmetry and similarity of equilateral triangles. Symmetry says $\triangle$ABC and $\triangle$PQR have the same centroid M. $\overline{\text{BC}}$ is twice as far (4 cm) from M as is $\overline{\text{QR}}$, so the linear scaling of PQR to ABC is 1:2. Thus, $\triangle$PQR has side length $4\sqrt{3}$ cm and enclosed area $12\sqrt{3}$ cm^2. P is at $(0, 4)$. We do not need Q and R; using symmetry it suffices to work out the details at P, and the other two vertices are just alike. The set of points 7 cm above $\overline{\text{BC}}$ are on the line $y = 3$ cm [purple line], running 1 cm below P. Two special points are where the line $y = 3$ intersects $\triangle$PQR; the line lops off from vertex P an equilateral triangle whose altitude is 1/6 that of $\triangle$PQR and area that is $(1/6)^2$ that of $\triangle$PQR, or $\frac{12\sqrt{3}}{36} = \frac{\sqrt{3}}{3}$ cm^2. With 3 such congruent triangles being lopped off (at P, Q, R), a total of $\sqrt{3}$ cm^2 is lopped off from $\triangle$PQR's area of $12\sqrt{3}$ cm^2. So, the area enclosed by the desired hexagon is $11\sqrt{3} = 19.052 \approx \mathbf{19.1}$ cm^2.