MATHCOUNTS®

2022 Chapter Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

The following pages provide detailed solutions to the Sprint and Target Rounds of the 2022 MATHCOUNTS Chapter Competition. These solutions show creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions with the MATHCOUNTS community.

Sprint 1

$3^4 - 2 \times 4^2 = 81 - 2 \times 16$ [Exponentiation first.]
$= 81 - 32$ [Multiplication next.]
$= \mathbf{49}$. [Subtraction last.]

Sprint 2

$$\begin{array}{r} 4\,326\,052 \\ -\underline{4\,325\,131} \\ \mathbf{921} \end{array}$$

Sprint 3

Reordering the set elements in increasing order yields {0, 3, 4, 7, 8, 11, 16}. Since we have an odd number of elements, the median is the middle element of this ordered set. In the case of this set, the middle element is the fourth element, which is **7**.

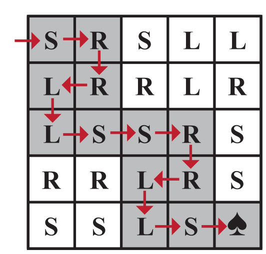
Sprint 4

Parker got 11. Each friend got twice that: $2 \times 11 = 22$. There are 3 friends, so the total count is $11 + 3 \times 22 = 11 + 66 = \mathbf{77}$ pieces.

Sprint 5

The iterative steps 3 and 4 yield what is called the digits-sum of the original number. Summing the digits of a number and taking the remainder upon dividing the sum of the digits by 9 gives the remainder upon dividing the original number by 9. The only nonnegative integer that yields a digit-sum of 0 is 0; a positive integer that is divisible by 9 has a digit-sum of 9, instead of the actual remainder value 0. The sum of the digits of 135 is 9, so its digit-sum is 9, indicating 135 to be divisible by 9. If we multiply a positive multiple of 9 by *any* positive integer (it does not have to be two-digit), the result is also a multiple of 9, so the digit-sum is **9**.

Sprint 6



Let's trace through the rooms, starting with the arrow entering the room at the upper left, shading each room that we enter, and drawing an arrow into the next room in accordance with the L, R, or S in the room we just entered. Once we reach the spade room, we count the number of shaded rooms, which is **13** rooms.

Sprint 7

$n + 18 = 4 \times 5 = 20$, so $n = 20 - 18 = \mathbf{2}$.

Sprint 8

We need an integer that is divisible by both 2 and 3, thus by 6, and it must be greater than 100. Since $100 \div 6 = 16$ R 4, the next multiple of 6 up from 100 is $17 \times 6 = 102$. So, the fewest number of students who could have signed up is **102** students.

Sprint 9

Rectangular prisms have 6 faces. For any 1 face, there are 4 adjacent faces that must not be colored the same as the first face, and only the 1 remaining face is non-adjacent and permitted to be the same color. Therefore, at most 2 faces (opposite faces) are allowed to be the same color, so at least $6/2 = 3$ colors are needed. There are 3 pairs of opposite sides, so the minimum number of colors needed is **3** colors.

Sprint 10
Let l be the length of the rectangle. Then the width $w = l + 1$. We need the perimeter, which is $2(l + w) = 2(l + l + 1) = 4l + 2$. We are given that 72 ft^2 is the enclosed area lw. Therefore, $72 = lw = l(l + 1) = l^2 + 1l$. Rearranging yields: $0 = l^2 + 1l - 72 = (l + 9)(l - 8)$, so $l = -9$ feet or $l = 8$ feet. It makes no sense for the length of a side to be negative, so we must have $l = 8$ feet and perimeter 4×8 ft $+ 2$ ft $= \mathbf{34}$ feet.

Sprint 11
Because there are $\frac{3}{8}$ as many heads as there are legs, there are $\frac{8}{3}$ as many legs as heads, so the average number of legs per head is $2\frac{2}{3}$, which is $\frac{2\frac{2}{3} - 2}{4 - 2} = \frac{2/3}{2} = \frac{1}{3}$ of the way from 2 [# legs per chicken] to 4 [# legs per goat]. Therefore, $\frac{1}{3}$ of the animals are goats, and $\frac{2}{3}$ are chickens, so the goat-to-chicken ratio is 1:2. Thus, the minimum number of animals is $1 + 2 = 3$ animals, meaning there is 1 goat and **2** chickens.

Sprint 12
The two highest scores are 8.5 and one of the two 8.0 scores, so these are discarded; the two lowest scores are 7.0 and 7.0 (both of the two 7.0 scores), so these are discarded. The scores that are used are the one remaining 8.0 and the two scores of 7.5. Therefore, the point total for the dive is $3.5(8.0 + 7.5 + 7.5) = 3.5(23) = 3(23) + \frac{1}{2}(23) = 69 + 11.5 = \mathbf{80.5}$ points.

Sprint 13
$1\frac{3}{4} = \frac{7}{4}$ cups are required for 24 cookies, so for 18 cookies: $\frac{18}{24}\left(\frac{7}{4}\right) = \frac{3}{4}\left(\frac{7}{4}\right) = \frac{21}{16} = \mathbf{1\frac{5}{16}}$ cups.

Sprint 14
With a rectangle, two vertices may be the endpoints of a short side, of a long side, or of a diagonal. Given two such lengths, the shorter must be a side, while the longer may be a side or a diagonal. If the 5 meters is a diagonal, then we have a 3-4-5 right triangle and the unknown length is the second side, which must be 4 meters. If the 5 meters is the longer side, the unknown length is the diagonal, even longer than the 5 meters. Thus, the minimum possible distance in question is **4** meters.

Sprint 15
Of the 64×144 calculators in the lot, 64×12 were tested. Therefore, the fraction tested is $\frac{\cancel{64\times}12}{\cancel{64\times}144} = \frac{1}{12}$. If 12 times as many calculators are tested, we expect 12 times as many failures, thus $12 \times 2 = \mathbf{24}$ calculators.

Sprint 16
This problem involves a weighted average with the \$30 value weighted as $75\% = \frac{3}{4}$ and the \$10 value weighted as $25\% = \frac{1}{4}$. So, $\frac{3}{4}(30) + \frac{1}{4}(10) = \frac{90+10}{4} = \frac{100}{4} = \$\mathbf{25}$ or $\$\mathbf{25.00}$.

Sprint 17
$$\left(11 + \frac{1}{2}\right) \times \left(12 + \frac{5}{12}\right) = \left(11 \times 12 + 11 \times \frac{5}{12} + \frac{1}{2} \times 12 + \frac{1}{2} \times \frac{5}{12}\right) = \left(132 + \frac{55}{12} + 6 + \frac{5}{24}\right) =$$
$$\left(132 + 6 + \frac{2\times 55 + 5}{24}\right) = \left(138 + \frac{115}{24}\right) = 142\frac{19}{24}\ \text{ft}^2,$$ which rounds to $\mathbf{143}$ ft^2.

Sprint 18
The length l and width w are related as $w = l - 50$. Therefore, the perimeter is given by $500 = 2(l + w) = 2(l + l - 50) = 4l - 100$, so $l = \frac{500 + 100}{4} = 150$ ft. The enclosed area is given by $lw = l(l - 50) = 150 \times 100 = \mathbf{15,000}$ ft^2.

Sprint 19
We need to determine the estimated arrival time down to the minute, so we need to determine the remaining travel time, distance divided by speed, in terms of minutes: $\frac{60 \cancel{\text{mi}}}{75 \frac{\cancel{\text{mi}}}{\cancel{\text{hour}}}} \times \frac{60 \text{ minutes}}{\cancel{1 \text{ hour}}} = \frac{4}{5} \times 60$ minutes $=$ 48 minutes, which is 12 minutes short of 1 hour. Therefore, starting at 10:32, add 1 hour to get 11:32, and then subtract 12 minutes to result in **11:20** a.m.

Sprint 20
Based on the inforamtion given, we have $x^5 = \frac{2}{3}x^4$, so $x^4\left(x - \frac{2}{3}\right) = 0$. Therefore, $x = 0$ or $x = \frac{2}{3}$, but it must be the latter value because 0 does not satisfy the positivity requirement. Thus, $\frac{x^{10}}{x^8} = x^2 = \left(\frac{2}{3}\right)^2 = \mathbf{\frac{4}{9}}$.

Sprint 21
If the two bread slices are selected to be the same, then there are 3 flavor choices. If they are selected to be different, then there are ${}_3C_2 = \frac{3!}{2!\,1!} = 3$ choices, totaling 6 choices for the bread. For each of these, there are 3 choices of filling (ham only, cheese only, both). So, James can make $6 \times 3 = \mathbf{18}$ different sandwiches.

Sprint 22
Let N be the total number of students in Mr. Short's homeroom. Then working backward, we see that $6 = \frac{1}{4}\left(1 - \frac{1}{3}\right)N = \frac{1}{4}\left(\frac{2}{3}\right)N = \frac{1}{6}N$, so $N = 6 \times 6 = \mathbf{36}$ students.

Sprint 23
We need meters per second, not centimeters per second, so convert the stride of 140 cm to meters: $140 \cancel{\text{cm}} \times \frac{1 \text{ m}}{100 \cancel{\text{cm}}} = 1.4$ meters. The number of pulses going above the dashed line for the 3 g threshold to count as strides is 19. Therefore, a total assumed distance of 19×1.4 meters [do *not* calculate that yet] is covered in 20 seconds, making the average speed $\frac{19 \times 1.4}{20} = 19 \times 0.07 = \mathbf{1.33}$ m/s. [Notice that waiting so we could do the very easy division first made for a simpler multiplication later, which can be important for the Sprint Round as a speed round without support of electronic calculators.]

Sprint 24
The triangle inequality property tells us $3 = 5 - 2 < x < 5 + 2 = 7$, so x as an integer must be 4, 5 or 6. The median is the middle value, **5**.

Sprint 25
\$150 each year averages to $\frac{\$150/\cancel{\text{yr}}}{12 \frac{\text{mo}}{\cancel{\text{yr}}}} = \frac{\cancel{6\times}\$25}{\cancel{6}\times 2 \text{ mo}} = \$12.50/\text{mo}$. So, $\$5.95N > \$12.50/\text{mo} + \$3.95N$, where N is the number of movies rented per month. Thus, $\$2N > \$12.50/\text{mo}$, making $N > \frac{\cancel{\$}12.50/\text{mo}}{\cancel{\$}2} = 6.25/\text{mo}$. The least such integer is **7** movies per month.

Sprint 26

Note that $345{,}600 = 3456 \times (2 \times 5)^2$. The sum of the digits of 3456 is 18, which is divisible by 9, so 3456 is likewise divisible by $9 = 3^2$, leaving a quotient of 384. The sum of the digits of 384 is divible by 3 but not by 9, so that is the case with $384 = 3 \times 128 = 3 \times 2^7$. Combining all this yields $345{,}600 = 2^9 \times 3^3 \times 5^2 = 2^6 \times (2^3 \times 3^3) \times 5^2 = 4^{6/2} \times 6^3 \times 5^2 = 6^3 \times 5^2 \times 4^3$. The product of the exponents is $abc = 3 \times 2 \times 3 = \mathbf{18}$.

Sprint 27

Of 12 flowers, 6 are required to be orchids, leaving flexibility for the remaining 6 to be any mix of 4 kinds of flowers, including possibly more orchids. Let's use what I call the bars and blanks method. [A variety of names are given to this method.] We have 12 blanks, each representing one flower, with O (for orchid) in the last 6 since we require a minimum of 6 orchids; we have 3 bars to intersperse among the empty blanks to separate the roses from the lilies, the lilies from the violets, and the violets from the extra orchids. For example: | ___ ___ ___ ___ || ___ ___ _O_ _O_ _O_ _O_ _O_ _O_ would indicate 0 roses (because of 0 blanks to the left of the first bar), 4 lilies (because of 4 blanks between the first and second bars), 0 violets (because of 0 blanks between the second and third bars) and 2 extra orchids (because of two empty blanks to the right of the last bar, for a total of 8 orchids). The number of possible distinct bouquets is the number of orderings of the 3 bars and the 6 empty blanks, which is ${}_{6+3}C_6 = \frac{9!}{6!\,3!} = \frac{9\times8\times7}{3\times2\times1} = \frac{72\times7}{6} = 12 \times 7 = 84$. So, that's **84** groups of a dozen flowers.

Sprint 28

Cross-multiplying yields $(x+2)(y+3) = 6$. The only ways to factor 6 into the product of two integers is $(1)(6)$, $(2)(3)$, $(-1)(-6)$ and $(-2)(-3)$—each of which can go in either order as to which factor is $(x+2)$ and which is $(y+3)$. The 8 factors for $(x+2)$ occur in 4 pairs, each pair having one positive value and one counterpart equal-magnitude negative value, with the sums canceling each other to 0, so the sum of all eight $(x+2)$ factors is 0. To get the sum of all eight corresponding x values, subtract 2 for each of the 8 factors to end up with $0 - 8 \times 2 = \mathbf{-16}$.

Sprint 29

$a_0 = 4$;
$a_1 = a_1$;   [an as yet unknown value]
$a_2 = a_1 + 2a_0 = a_1 + 8$;
$a_3 = a_2 + 2a_1 = a_1 + 8 + 2a_1 = 3a_1 + 8$;
$a_4 = a_3 + 2a_2 = 3a_1 + 8 + 2(a_1 + 8) = 5a_1 + 24 = 26$, so $a_1 = \frac{26-24}{5} = \frac{2}{5}$ and $a_3 = \frac{6}{5} + 8 = \frac{46}{5}$;
$a_5 = a_4 + 2a_3 = 26 + 2 \times \frac{46}{5} = \frac{130+92}{5} = \mathbf{\frac{222}{5}}$.

Sprint 30

Each of the 8 vertex blocks has 3 sides painted, thus probability $\frac{1}{2}$ coming up unpainted. The 12 edge blocks except vertex blocks total $12 \times 8 = 96$ blocks, each having 2 sides painted, thus probability $\frac{2}{3}$ coming up unpainted. The 6 face blocks except the edge and vertex blocks total $6 \times 8 \times 8 = 384$ blocks, each having 1 side painted, thus probability $\frac{5}{6}$ coming up unpainted. Each of the remaining blocks, the interior $8 \times 8 \times 8 = 512$ blocks, has 0 sides painted, thus probability 1 coming up unpainted. Therefore, the overall probability of all blocks rolling with the upward facing side being unpainted is given by $\left(\frac{1}{2}\right)^8 \left(\frac{2}{3}\right)^{96} \left(\frac{5}{6}\right)^{384} 1^{512} = \frac{2^{96}\times5^{384}}{2^8\times3^{96}\times2^{384}\times3^{384}} = 2^{-296} \times 3^{-480} \times 5^{384}$, so the desired answer is the sum of the exponents, $-296 - 480 + 384 = \mathbf{-392}$.

Target 1

The excluded value is the product of all 6 values divided by the given product of all included values: $\frac{1\times4\times6\times9\times15\times17}{3240} = \frac{55{,}080}{3240} = \mathbf{17}$.

Target 2

$\frac{1\times8+1\times11+2\times12+4\times14+6\times15+7\times16+2\times17+3\times18+4\times20}{1+1+2+4+6+7+2+3+4} = \frac{8+11+24+56+90+112+34+54+80}{30} = \frac{469}{30} = 15.63\ldots$, which rounds to the nearest tenth as **15.6** points.

Target 3

$\frac{88\ \cancel{\text{measure}} \times 16\ \frac{\cancel{\text{notes}}}{\cancel{\text{measure}}}}{432\ \frac{\cancel{\text{notes}}}{\cancel{\text{min}}}} \times \frac{60\text{ seconds}}{1\ \cancel{\text{min}}} = \frac{84{,}480}{432} \approx \mathbf{196}$ seconds.

Target 4

Because the diameter of the circle is 8, radii OA, OB and OC have length 4. Given $BC = 4\sqrt{2}$, we have $(OB)^2 + (OC)^2 = 32 = (BC)^2$, which, according to the Pythagorean theorem, means that $\triangle BCO$ is a right triangle with angle BOC being the right angle and the two radii being the legs. Therefore, the area enclosed by triangle BOC is $\frac{1}{2} \times 4 \times 4 = 8$. Angles BOC and AOC are supplementary, so angle AOC is a right angle as well, and the sector formed by minor arc AC and radii OA and OC is one quadrant of the circle. Therefore, the area enclosed by the sector is $\frac{1}{4}\pi(4^2) = 4\pi$. The total area of the shaded area is $8 + 4\pi$. So, $a = 8$, $b = 4$, and $ab = 8 \times 4 = \mathbf{32}$.

Target 5

$\frac{90^2}{43{,}560} \times 100\% = \frac{810{,}000}{43{,}560}\% \approx \mathbf{18.6}\%$.

Target 6

Both PQ and PR are the hypotenuse of $2 \times$ 3-4-5 right triangles. By properties of 45-45-90 right triangles, $QR = 6\sqrt{2}$. Thus, triangle PQR is an isosceles triangle with side lengths 10, 10 and $6\sqrt{2}$. To find the enclosed area, we can easily split the triangle into two congruent right triangles with hypotenuse 10 and one leg of length $\frac{1}{2} \times 6\sqrt{2} = 3\sqrt{2}$, so the other leg is $\sqrt{10^2 - (3\sqrt{2})^2} = \sqrt{100 - 18} = \sqrt{82}$ and the enclosed area is $\cancel{2} \times \frac{1}{\cancel{2}} \times 3\sqrt{2} \times \sqrt{82} = \mathbf{6\sqrt{41}}$ units^2.

Target 7

$8 \times (22 \times 16 + 14 \times 4) \times 54 = 8 \times (352 + 56) \times 54 = 408 \times 432 = 176\,256$, which when rounded to the nearest thousand is **176,000** books.

Target 8

To have the product of the 4 digits be divisible by 14, the digit 7 must be chosen at least once, and one or more of the digits 2, 4, 6 and 8 must be chosen at least once. This is the complement of never choosing 7, which has probability $\left(\frac{8}{9}\right)^4 = \frac{4096}{6561}$ or never choosing an even digit, which has probability $\left(\frac{5}{9}\right)^4 = \frac{625}{6561}$. However, these two cases overlap (choosing 1, 3, 5 or 9 every time), which has probability $\left(\frac{4}{9}\right)^4 = \frac{256}{6561}$. Thus, the answer is $1 - \left(\frac{4096}{6561} + \frac{625}{6561} - \frac{256}{6561}\right) = \mathbf{\frac{2096}{6561}}$.

Team 1

$75\% = 0.75 = \frac{3}{4}$. Now, $\frac{2}{3} \times 75\% \times 0.85x = \frac{2}{3} \times \frac{3}{4} \times 0.85x = \frac{1}{2} \times 0.85x = 0.425x = 100$, so $\frac{4}{5} \times 70\% \times 0.75x = \frac{\cancel{4}}{5} \times \frac{7}{10} \times \frac{3}{\cancel{4}}x = \frac{21}{50}x = 0.42x = 0.42 \times \frac{0.425}{0.425}x = \frac{0.420}{0.425} \times 0.425x = \frac{420}{425} \times 100 = 98.82\ldots$, which rounds to the nearest tenth as **98.8**.

Team 2

The count in the category "Insects" is 3252; the count in all other categories combined, 1892, is less than for the one "Insects" category. Whenever one category has a count that is more than half the total count (that is, the one category has a greater count than all other categories combined), the median value is guaranteed to occur in that one category. All members of that one category in this case have a value of 6, so the median number of legs is **6** legs.

Team 3

The number being divisible by both 2 and 5 means the ones digit must be 0. To be divisible by 3, the sum of the digits must be divisible by 3: $2 + 3 + 4 + h + 6 + 0 = 15 + h$, so the hundreds digit must be divisible by 3, thus 0, 3, 6 or 9. The largest of these, 9, yields the greatest possible overall value for the original number, **234960**.

Team 4

Ailey, and therefore Zander, earn $\frac{47.25}{3}$, which is \$15.75 per hour, so Liz earns $\frac{4}{5} \times 15.75 = \$12.60$ *per* hour. Ailey and Zander worked a combined $3 + 5 = 8$ hours, while Liz worked 12 hours. Therefore, the combined earnings are $8\ \cancel{\text{hours}} \times \frac{\$15.75}{\cancel{\text{hour}}} + 12\ \cancel{\text{hours}} \times \frac{\$12.60}{\cancel{\text{hour}}} = \$126 + \$151.20 = \$277.20$. Of that, $5\% = \frac{1}{20}$ went back into the business, thus $\frac{277.20}{20} = \$\mathbf{13.86}$.

Team 5

Each side of the triangle is the sum of the two radii of the circles forming the side:
$1 + 2 = 3$ meters; $1 + 3 = 4$ meters; $2 + 3 = 5$ meters. Therefore, we have a 3-4-5 right triangle whose enclosed area is half the product of the two shorter sides: $\frac{1}{2} \times 3 \times 4 = \mathbf{6}\ \text{m}^2$.

Team 6

For a page number $100h + 10t + u \leq 280$, h, t and u represent the hundreds, tens, and units or ones digits, respectively, each at least 0 and at most 9, subject to further constraints, two of which are $h + t + u = 16$ and $h \leq 2$. When $h = 0$, $t + u = 16$, so t can range from 7 up to 9, while u ranges from 9 down to 7—making 3 options. When $h = 1$, $t + u = 15$, so t can range from 6 up to 9, while u ranges from 9 down to 6—making 4 options. Careful now: When $h = 2$, $t + u = 14$, so t can range from 5 up to only 7 [due to a page number being at most 280], while u ranges from 9 down to 7—making 3 options. Therefore, the total number of words written is $3 + 4 + 3 = \mathbf{10}$ words.

Team 7

There are $5! = 120$ permutations of the five digits. Each digit value occurs $\frac{1}{5}$ of the time in each digit place. Therefore, each of 1, 2, 3, 4 and 5 occurs 24 times in each digit place, so the sum of the digit values in any one digit-place column is $24(1 + 2 + 3 + 4 + 5) = 24 \times 15 = 360$. To account for the 10,000s place, the 1000s place, the 100s place, the 10s place and the 1s place, we need to multiply the 360 by $10{,}000 + 1000 + 100 + 10 + 1 = 11{,}111$ to end up with the answer $11{,}111 \times 360 = \mathbf{3{,}999{,}960}$.

Team 8

For a regular n-gon, there are always $\frac{n(n-3)}{2}$ diagonals, so for $n = 20$, there are 170 diagonals. When n is even, the longest diagonals go straight across passing through the center to the opposite vertex, and there are 10 such distinct pairings of vertices. The shortest diagonals are obtained by taking any vertex; that vertex is adjacent to two other vertices, and the diagonal joining those two other vertices has the shortest possible length—there are 20 such vertices and 20 such diagonals. Therefore, of the 170 diagonals, we are rejecting $10 + 20 = 30$ and thus keeping 140. The probability is, therefore, $\frac{140}{170} = \mathbf{\frac{14}{17}}$.

Team 9

Line 1: either '2' or '3' first, so 2 orderings; 4 choices for the '2'; 3 choices for the '3';
thus, $2 \times 4 \times 3 = 24$ distinct lines.
Line 2: '3' in any of 3 positions, so 3 orderings; 3×2 choices for the '2's; 2 choices for the '3';
thus, $3 \times 3 \times 2 \times 2 = 36$ distinct lines.
Line 3: either '2' or '3' first, so 2 orderings; 1 choice left for the '2'; 1 choice left for the '3';
thus, $2 \times 1 \times 1 = 2$ distinct lines.
Putting the options together for all three lines: $24 \times 36 \times 2 = \mathbf{1728}$ haiku.

Team 10

The square root causes major difficulties for having an integer result. It is necessary to have $\sqrt{ab} = \sqrt{40b} = 2\sqrt{10b}$ be an integer, so $10b$ must be a perfect square, so it must be that $b = 10n^2$ for some integer $n > 2$. If n is divisible by 3, then the second and third terms in the numerator $40 + 20n + 10n^2$ are divisible by 3 but the first term is not, so the numerator cannot be divisible by 3. We can crank a few cases quickly, so let's not analyze more. Because $n > 3$, first try $n = 4$: $40 + 80 + 160 = 280$—not divisible by 3. Next try $n = 5$: $40 + 100 + 250 = 390$—yes, divisible by 3. Therefore, the least $b > 2$ is $b = 10 \times 5^2 = \mathbf{250}$.