2005-2006: Think Outside the Cube: How many distinct cubes are possible?

Each face of the cube must be painted either white or red, but there are no other restrictions. We also are told that each face of the cube is identical. This means that it is not like working with a die in which the six faces are distinguishable. For our cube, if we decide to paint exactly one face red, it will not matter which face we choose. Each choice would lead to the same cube. So there is only **one way** to create a cube with exactly one face painted red. Things change when we paint exactly two faces red. We could have a cube with two adjacent faces painted red or two parallel faces painted red. These are the only **two scenarios** that would give us different cubes (Figure 1). What if we paint exactly three faces red? These three faces could be the three faces sharing a vertex (all connected at a corner) or three faces that wrap around the cube consecutively. These are the only **two possibilities** (Figure 2). What if four faces are to be painted red? We have already figured this out. This would mean exactly two faces are painted white. We already know there are exactly **two ways** to paint two faces red, so there are also only two ways to paint two faces white. Likewise, painting five faces red will mean painting one face white, and we know there is exactly **one way** to do that. The only other **two options** are painting none of the faces red (meaning they are all white) and painting all of the faces red. This is 1 + 2 + 2 + 2 + 1 + 2 = 10 distinct cubes with only white and red faces.

Figure 1

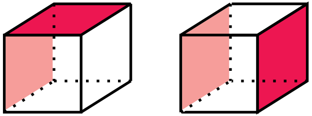

Figure 2