MATHCOUNTS®

2020 School Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2020 MATHCOUNTS® School Competition. These solutions provide creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

2020 School Sprint Round Solutions

1. The right end of the washi tape lies at 24 cm, and the left end of the washi tape lies at 12 cm. Taking the difference gives 24 − 12 = **12** cm.

2. The first step in the addition is to compute the units digits, we have 5 + 9 = 14. We will put a 4 in the units digit of our sum and carry the 1 to the tens digits addition to get 1 + 4 + 8 = 13. This means the tens digit will be a **3**.

$$\begin{array}{r} {\scriptstyle 1\,1\;\;} \\ 12{,}345 \\ +\,56{,}789 \\ \hline 34 \end{array}$$

3. We know that Dora drinks 12 ounces of cola. If Erika drinks half of a 20-ounce bottle, then Erika drinks (1/2)(20) = 10 ounces of cola. So, Dora drinks 12 − 10 = **2** more ounces of cola than Erika.

4. If we subtract, we get 1,000,000 − 128,000 = **872,000**.

5. The area of a rectangle is calculated using the formula area = $L \times W$, where L = length and W = width. We can start by substituting the values that we know into this equation, which gives 105 = $L \times 7$. Dividing both sides of the equation by 7 to solve for L, we get L = **15** inches.

6. The value of π is often approximated in decimal form as 3.14, but another common approximation in fraction form is 22/7. The latter will be our best choice for this solution. We then find that −7π ≈ −7 × 22/7 ≈ **−22**.

7. Starting at −17 degrees, we are told the temperature increases by 23 degrees, so −17 + 23 = 6 degrees. Then, we are told that the temperature decreases by 5 degrees, so 6 − 5 = **1** degree.

8. The total number of lunch orders on Friday is 132 + 112 + 88 + 44 + 24 = 400. The percent of orders that were spaghetti is 44/400 = 11/100 = **11**%.

9. If we start to list out the positive odd integers, and number them as we go, we get:

   Positive odd integers: 1 3 5 7 9....
   Number in the sequence of positive odd integers: *1 2 3 4 5*....

   You'll notice that there is a pattern here relating the integers and their number in the sequence. We can set this up as an equation: Integer = (number in sequence × 2) – 1. So, if we're looking for the 40th number in the sequence, the integer in question would be (40 × 2) − 1 = **79**.

10. First we compute the denominator value of the second term: 1 − 2/3 = 1/3. We then have 1 + 2/(1/3). In the second term, we have division by a fraction which is the same as multiplication by the fraction's inverse. We can rewrite this and solve: 1 + 2 × 3 = 1 + 6 = **7**.

11. To simplify $(x^2y^3)^2$, we can use the Power of a Product Rule for Exponents, which says that to simplify an expression such as this, the exponent outside of the parentheses should be distributed to each of the elements being multiplied together within the parentheses. So, $(x^2y^3)^2$ would become $(x^2)^2(y^3)^2$. Then, by the Power Rule for Exponents, we know that raising an exponential expression to an exponent means we multiply the exponents in the expression together. So, $(x^2)^2 = x^4$ and $(y^3)^2 = y^6$, so the whole expression simplifies to x^4y^6. Therefore, a + b = 4 + 6 = **10**.

    Alternatively, using the definition of exponents, you can expand the expression as follows: $(x^2y^3)^2 = (xxyyy)(xxyyy)$. Because multiplication is commutative, this can be rearranged as $xxxxyyyyyy = x^4y^6$.

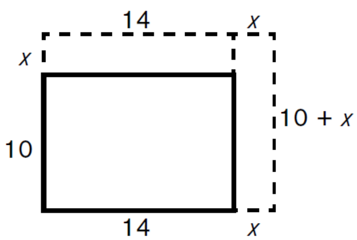
12. Kalyani won when she chose rock and Mark chose scissors, when she chose paper and Mark chose rock and when she chose scissors and Mark chose paper, as shown by the circled numbers in the table. In total, Kalyani won 20 + 24 + 21 = **65** matches.

13. With 5,280 feet in 1 mile and 12 inches in 1 foot, there are 5,280 × 12 = 63,360 inches in 1 mile. So, there are 63,360 ÷ 6 = **10,560** inches in one-sixth of a mile.

14. In 2000, the sum of their ages was 0 + 25 + 30 = 55. Every year that passes will add 3 years to the sum of their ages since each of them will age one year. We can set up the equation $55 + 3x = 100$. Solving for x, we find that $x = (100 - 55)/3 = 45/3 = 15$. This means the sum of their ages will be 100 in the year 2000 + 15 = **2015**.

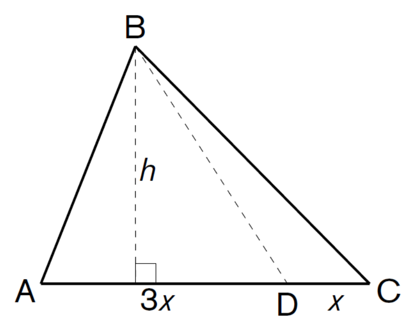
15. The dollar amount the price was reduced is \$200 − \$140 = \$60. So, we want to know what percent \$60 is of \$200. We can find this by dividing 60/200 = 0.3, and then multiplying by 100. So, 0.3 × 100 = **30**%.

16. One way to solve this is geometrically. We can draw the figure shown at the left and call the unknown distance we extend each side x. Using this figure, we can set up the equation $14x + x(10 + x) = 81$ or $x^2 + 24x - 81 = 0$. We can factor this equation into two binomials and write it as $(x - 3)(x + 27)$. The two solutions to this equation are −27 and 3. The solution x =3 is the only one that makes sense in this context, therefore, we know the side lengths were extend by 3 feet each and our largest side will be 14 + 3 = **17** feet.



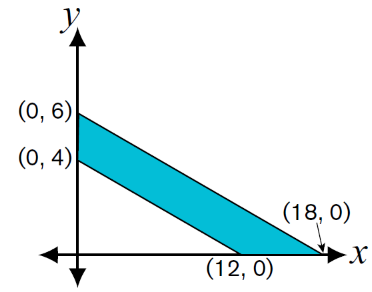
Alternatively, you could solve numerically. The pen had an original area of 10 × 14 = 140 ft^2. With the increase, the total is 140 + 81 = 221 ft^2. Factoring, we can say 221 = 13 × 17, which is an increase of 3 feet from each of the original dimensions. The larger dimension of the new pen is **17** feet.17. We'll need to look at the differences in the sons' ages in order to determine their ages when Alfred is 37. The difference in Alfred's and Beto's ages is 11 – 9 = 2 years. So, when Alfred is 37, Beto is 37 – 2 = 35. The difference in Alfred's and Dell's ages is 11 − 4 = 7 years. So, when Alfred is 37, Dell is 37 − 7 = 30. The average of their ages when Alfred is 37 is (37 + 35 + 30)/3 = **34** years.

18. Let's draw $\triangle$ABC oriented with side AC as the base, shown to the right. We don't know the length of AC, but we know that AD is three times DC so let's label these lengths $3x$ and x. If we measure the height of $\triangle$ABC and $\triangle$BDC as the perpendicular distance from side AC to point B, they have the same height, h. Using these, we can write the equation for area of $\triangle$ABC and $\triangle$BDC as $(1/2)(4x)(h)$ and $(1/2)(x)(h)$, respectively. We see that the area of $\triangle$BDC is $[(1/2)(x)(h)]/[(1/2)(4x)(h)] = 1/4$ the area of $\triangle$ABC, which we have been given. So we can conclude that the area of $\triangle$BDC is (1/4)(40) = **10** units².



19. We can use an equation to represent the provided scenario, $(d + 15) = 4(c + 15)$, where d = the number of people who own a dog and c = the number of people who own a cat. We can simplify this equation as follows: $d + 15 = 4c + 60 \rightarrow d = 4c + 45$. We also know that Kendra surveyed a total of 100 people, so $c + d + 15 = 100$. If we substitute $4c + 45$ in for d, we get: $c + (4c + 45) + 15 = 100 \rightarrow 5c + 60 = 100 \rightarrow 5c = 40 \rightarrow c = 8$. So, if we take the 8 people who own only a cat and the 15 people who own both a cat and a dog, we get 8 + 15 = **23** people who own a cat.

20. Let's first find the intersection of each of the two lines with the x- and y-axes. When $x = 0$, the lines will intersect the y-axis at $y = 4$ and $y = 6$. When $y = 0$, they will intersect the x-axis at $x = 12$ and $x = 18$, respectively. We can sketch this in the first quadrant as shown. We can think of each of the two lines as forming a right triangle with the x- and y-axis and find the shaded area by subtracting the smaller one from the larger one. Setting this up and solving, we get (1/2)(18)(6) − (1/2)(12)(4) = 9(6) − 6(4) = 6(9 − 4) = 6(5) = **30** units^2.



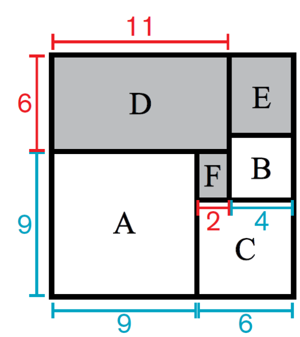
21. The first positive 100 integers are 1, 2, 3, 4, 5, ... , 98, 99, 100. We know that the median must fall exactly in the middle of the increasing arranged numbers, so it will fall between 50 and 51. We can find the median by averaging 50 and 51: (50 + 51)/2 = 50.5. In order to find the mean, we must first add up all of the positive integers between 1 and 100. Let's start by lining up the numbers 1 to 50 in increasing order, matched up with the numbers 51 to 100 in decreasing order. See below:

1	2	3	4	5	6	7	8	9	10	.....	40	41	42	43	44	45	46	47	48	49	50
100	99	98	97	96	95	94	93	92	91	....	61	60	59	58	57	56	55	54	53	52	51

You'll notice that each column sums to 101. For example, 1 + 100 = 101; 2 + 99 = 101; 3 + 98 = 101; etc. So, since we have 50 of these sums, we can calculate 101 × 50 = 5050 to give the sum of the positive integers from 1 through 100. To find the mean, 5050/100 = 50.5. So, by taking the difference between the mean and median of 50.5 − 50.5 = 0, we get an absolute difference of **0**.

Alternatively, you could get the final answer without any calculation. Because we are looking at the first 100 positive integers, we know that the distribution of numbers is symmetric which means the mean and median will be the same and therefore have and absolute difference of **0**.

22. Since the area of square A is 81 units2, the area of square B is 16 units2 and the area of square C is 36 units^2, we know the side lengths of squares A, B and C, respectively, are 9, 4 and 6 units. The width of rectangle F with be the length of the side of square C minus the length of the side of square B or 6 − 4 = 2 units. With this information, we can now find the dimensions of rectangle D. Its width will be the length of side A plus the width of rectangle F or 9 + 2 = 11 units. Since the entire figure is a square, we know that all sides of will be 9 + 6 units in length, which means that the height of rectangle D must be 6 units. The area of rectangle D is11 × 6 = **66** units^2.



23. The total number of possibilities for arrangements of how the coins land over four tosses is 2 × 2 × 2 × 2 = 16. This is because there are two possibilities (heads or tails) for each toss of the coin. Of those 16 possible arrangements, we want to know how many have at least 3 heads (i.e. how many have 3 or 4 heads). Those arrangements are: HHHH, HHHT, HHTH, HTHH, and THHH. So, there are 5 arrangements with at least 3 heads, and thus the probability of this happening is **5/16**.

24. The prime factorization of 126 is 2 × 32 × 7. Since this is the least common multiple of n and 9 = 32, we know n has to have 2 and 7 as factors. Suppose these were the only factors and n = 2 × 7 = 14, this would not satisfy the condition that the greatest common factor of n and 18 is 6. The GCF(14, 18) = 2 since 14 = 2 × 7 and 18 = 2 × 32. To make it so the GCF(n, 18) = 6 or 2 × 3, we need to add an additional factor of 3 to n. This makes our n = 2 × 3 × 7 = **42**.

25. We know the lines intersect on the x-axis, which means the y-coordinate of this intersection point is equal to 0. So, let's substitute this into the equation y = 9 − $1/3x$ and solve for the value of x. We get 0 = 9 − $1/3x$ → −9 = −$1/3x$ → 27 = x. In addition, we know that the slopes of perpendicular lines are opposite reciprocals of each other. So, since the provided slope is −1/3, the slope of the other line, $y = mx + b$, must be 3. Now, let's substitute everything we know into the equation $y = mx + b$ and solve for b. We get 0 = 3(27) + b → 0 = 81 + b → −81 = b. Therefore, b = **−81**.

26. There are two scenarios in which a product of two standard dice is a perfect square: (1) Double of the same number are rolled. (2) A 4 and a 1 are rolled. When rolling two 6-sided dice, there are 6 × 6 = 36 possible rolls. Of these, there are 6 possible double rolls—each of the 6 possible numbers on the dice are rolled together. There are 2 possible rolls of a 4 and a 1—the first die is a 4 and the second die is a 1 and vice versa. This gives us a probability of (6 + 2)/36 = 8/36 = **2/9**.

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Alternatively, if we can chart all the possibilities by create the table shown to the right. Across the top are all the possible rolls of die 1 and down the left side are all the possible rolls for die 2. Filling in the products and highlighting the perfect squares, we find there is a probability of 8/36 = **2/9**.

27. The best way to organize the information in this problem is to create a table, as seen below. Let's start with the information most obvious to us - the Percent Juice column. We are told that the original punch contains 50% juice, the punch that is added contains 100% juice, and the result is a punch with 65% juice. So, we can fill these percents in in decimal form in column 2: 0.5, 1.0, and 0.65. Next, let's look at column 1, Gallons. Say x = the number of gallons of 50% juice and y = the number of gallons of 100% juice. We can add these into the corresponding cells. We know that $x + y = 2$ gallons. So, this information can go in the Mixture row of the Gallons column. Finally, we can multiply the number of gallons of each percent juice by the percent juice to get the total gallons of juice, which can be found below in column 3.

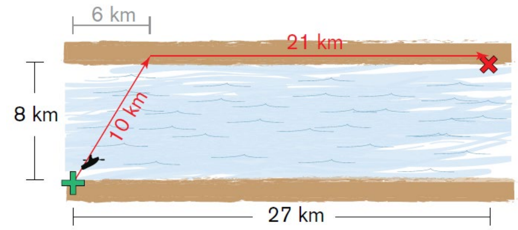
	Gallons	**Percent Juice (in decimal form)**	**Total Gallons of Juice**
50% Juice	x	0.5	$0.5x$
100% Juice	y	1.0	$1y$
Mixture	$x + y = 2$	0.65	(0.65)(2) = 1.30

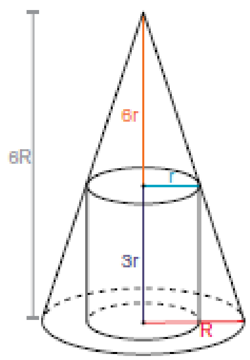
We know that we're going to need to solve for y - the amount of the mixture that was poured out is the same as the amount of the 100% juice with which it was replaced. Therefore, we can create an equation using column 3, Total Gallons of Juice: $0.5x$ + $1y$ = 1.30. However, with two unknowns, we'll need to do some manipulation before we can solve for y. First, if we know $x + y = 2$, we can subtract y from both sides of the equation to get $x = 2 - y$. We can substitute x in the Gallons column with $2 - y$ in our table, as seen below. Similarly, we can rewrite our total gallons of 50% juice as $0.5(2 - y)$, as seen in the Total Gallons of Juice column below.

	Gallons	**Percent Juice (in decimal form)**	**Total Gallons of Juice**
50% Juice	$2 - y$	0.5	$0.5(2 - y)$
100% Juice	y	1.0	$1y$
Mixture	$x + y = 2$	0.65	1.30

Finally, we can rewrite our equation in terms of y and solve for y: $0.5(2 - y)$ + $1y$ = 1.30 → 1 − 0.5 y + $1y$ = 1.30 → 1 + $0.5y$ = 1.30 → $0.5y$ = 0.30 → y = 3/5. Therefore, we would need 3/5 of a gallon of 100% juice to create the 65% juice mixture, which tells us that Gracie poured **3/5** of a gallon of the original mixture out.

28. Since Ursula rows in a straight line to a point 6 km downshore, we notice this forms a right triangle with legs of 8 km and 6 km. The hypotenuse or distance Ursula rowed is therefore 10 km. We know this without solving the Pythagorean Theorem because the triangle is a multiple of the 3-4-5 Pythagorean Triple. The remaining distance Ursula travels by running is 27 − 6 = 21 km. The total time to reach her distance is therefore 10 km ÷ 5 km/h + 21 km ÷ 14 km/h = 2 hr + 1½ hr = **3½** hours.



29. 

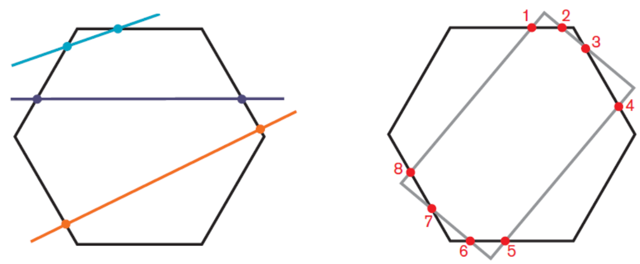
Take a look at the figure. It is important to understand that the cylinder is inscribed, so the space within the larger cone that is above the top of the cylinder creates a smaller cone that is proportional to the larger cone. Knowing this, let's break down the dimensions shown in the image. From the provided information in the problem, we know that the height of the larger cone is 6*R*, where *R* = the radius of the larger cone. Because the smaller cone above the cylinder is proportional to the larger cone, we know that the height of the smaller cone is 6*r*, where *r* = the radius of the smaller cone. This radius, *r*, is also the radius of the base of the cylinder, since the base of the cylinder is also the base of the smaller cone. Finally, from provided information in the problem, we know that the height of the cylinder is 3*r*. From this, we can derive that the height of the larger cone in terms of *r* is 6*r* + 3*r* = 9*r*. In order to use the volume formulas, we'll also need to know the value of *R* in terms of *r*. So, if we know that 9*r*:6*R*, we can reduce this ratio to be 3/2*r*:1*R*, so *R* = 3/2*r*. Using this, let's solve for the volume of the cone in terms of *r*. We get $V = \pi R^2(h/3) \rightarrow V = \pi(3/2r)^2(9r/3) \rightarrow V = \pi(9/4)r^2(3r) \rightarrow V = \pi(27/4)r^3$. Next, we'll need to find the volume of the inscribed cylinder. We get $V = \pi r^2 h \rightarrow V = \pi r^2 3r \rightarrow V = \pi 3r^3$. Now, we can compare the volumes of the two figures: Volume of cylinder/Volume of larger cone = $(3\pi r^3)/((27/4)\pi r^3) \rightarrow 3/(27/4) = 4/9$. This tells us that **4/9** of the larger cone's volume lies inside the cylinder.

30. Suppose the value we choose *x* to be is 25 which is the median value of the set {20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 210}. Each absolute value term in the sum measures a distance from *x*. With *x* = 25, the middle term, $|x - 25|$, will be 0. There will be 10 non-zero values left in the sum. Let's think of these 10 non-zero values as a measure of distance from this middle point—five values are to the left of our chosen *x* and five values are to the right. If we increase our *x* by 1 to be 25 + 1, we add 1 to all of the distances to the left, subtract 1 from all of the distances to the right and added 1 to our middle term which is no longer zero. This leads to a net change in the sum of +1. Similarly, if we decreased our *x* by 1 to be 25 − 1, we would add 1 to all of the distances to the right, subtract 1 from all of our distances to the left and add 1 to our middle term which is no longer zero. Again, for a net change of +1. We can conclude that any shift in *x* from this median value will result in an increase in the value of the sum. The *x* that minimizes the sum is therefore $2^5 = 32$. Note: It can be stated generally, that for any set of *S* of real numbers, is minimized by *x* equal to the median of the set. The set values chosen for this problem are arbitrary.

2020 School Target Round Solutions

1. In order to calculate the total amount of money Charles earned, we'll need to take the number of hires and multiply it by the amount earned per hire for each chore. Clothes ironing earnings = 9.50 × 10 = \$95. Dog walking earnings = 7.50 × 12 = \$90. Kitchen cleaning earnings = 12.00 × 7 = \$84. Snow shoveling earnings = 14.00 × 5 = \$70. So, in the month of January, Charlie's Chores earned a total of \$95 + \$90 + \$84 + \$70 = \$**339**.

2. Let's take a look at just one side of the quadrilateral—a straight line. Since we are told that no side of the quadrilateral lies on the same line as a side of the hexagon, the maximum number of times a side of the quadrilateral could intersect the hexagon is 2. You can use a ruler to test this on the image of the hexagon. There is no way to pass a straight line through the hexagon and have it intersect the shape more than 2 times. Therefore, the maximum number of points at which the quadrilateral could intersect the hexagon would be at 2 points per side, at 2 × 4 = **8** points.

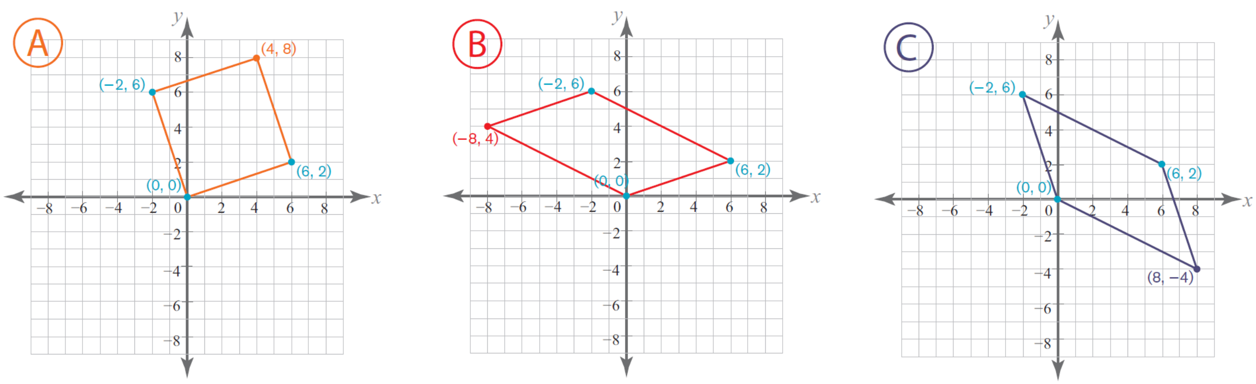


3. Express each of Penelope's and George's amount of candy in an equation. Let d = the number of days. So, Penelope's candy = 152 − 5d and George's candy = 124 − 4d. Since we want to know how many days it will take for them to have the same amount of candy, we can set the equations for Penelope and George equal to each other and solve for d, the number of days. We get $152 - 5d = 124 - 4d \rightarrow 152 = 124 + 1d \rightarrow 28 = d$. So, after **28** days George and Penelope will have the same amount of candy.

4. We're being asked to find the total number of bushels, and we are given the number of bushels grown per acre. We are also given the square footage of an acre, so we'll first need to find the area of the triangular field in square feet. For a triangle with base length b and height h, $Area = 1/2bh = 1/2(1320)(660) = 435{,}600$ ft^2. If 1 acre = 43,560 ft.^{2}, then there are 435,600 ft.^{2} / 43,560 ft.^{2} = 10 acres. Finally, knowing there are 160 bushels grown per acre, Rosie grows a total of 160 × 10 = **1,600** bushels.

5. Rolling a six-sided die five times produces 6^5 = 7,776 possible combinations of rolls, since there are 6 possible results in each of the five rolls. We're being asked to look specifically at the situations where all of the rolls are the same and where all of the rolls are different. We can deduce that there are exactly 6 scenarios where all of the rolls are the same - 11111, 22222, 33333, 44444, 55555, and 66666. To represent the scenarios where all of the rolls are different, we can use 6! This shows that on the first roll, the result could be any of the 6 numbers on the die. However, since all of the rolls must produce different results, there are now only 5 possibilities for the second roll in order to satisfy this requirement. By the same logic, there are only 4 possibilities for the third roll, and so on. To simplify, 6! = 6 × 5 × 4 × 3 × 2 = 720 possibilities. Since there are only 5 rolls in this scenario, we do not multiply by 1 at the end of this factorial expansion. However, we achieve the same result either way in this case, so we do not need to modify our 6! result in any way. Therefore, there are 720 + 6 = 726 possible combinations of rolls where either all are the same or all are different. So, the probability of this happening is 726 / 7776, which is approximately equal to **9.34**%.

6. As you can see in the images, there are three possible parallelograms that can be created with the three provided vertices. In image A, we find the slope between (0, 0) and (−2, 6) to be (6 − 0) / (−2 − 0) = 6 / −2 = −3. Because we must create a parallelogram, we know that the missing vertex must create a line segment that is parallel to the line segment between (0, 0) and (−2, 6), which means it must have the same slope. So, working up from (6, 2), using the slope of −3, we find the missing vertex to be (4, 8). So, a + b = 4 + 8 = 12. In image B, we find the slope between (6, 2) and (−2, 6) to be (6 − 2) / (−2 − 6) = 4/(−8) = −1/2. Because we must create a parallelogram, we know that the missing vertex must create a line segment that is parallel to the line segment between (6, 2) and (−2, 6), which means it must have the same slope. So, working up from (0, 0), using the slope of −1/2, we find the missing vertex to be (−8, 4). So, a + b = −8 + 4 = −4. In image C, as with image B, we find the slope between (6, 2) and (−2, 6) to be −1/2. Again, we know that the missing vertex must create a line segment that is parallel to the line segment between (6, 2) and (−2, 6), which means it must have the same slope. So, working down from (0, 0), using the slope of −1/2, we find the missing vertex to be (8, −4). So, a + b = 8 + −4 = 4. We can see from the above work that the least possible value for $a + b$ is **−4**.



7. Let's say Alan is the oldest, Ben is the second-oldest, and Craig is the youngest of the three. Let's also say that A = Alan, B = Ben, C = Craig, and the other two children are labeled 1 and 2. We're looking for all of the scenarios in which A finishes before B, who finishes before C. Let's start by looking at the scenarios where A, B, and C finish in adjacent places. This would mean we're looking for all of the possible arrangements of (ABC), 1, and 2. This can be represented by 3!, which simplifies to 3 × 2 × 1 = 6. Next, let's look at the scenarios where none of A, B, and C are in adjacent places. There are only 2 possible orders that satisfy this scenario: A1B2C and A2B1C. Finally, we need to consider the scenarios in which two of the three boys finish in adjacent places. There are 6 scenarios in which A and B finish in adjacent places: AB1C2, AB2C1, AB12C, AB21C, 1AB2C, and 2AB1C. Additionally, there are 6 scenarios in which B and C finish in adjacent places: A12BC, A21BC, A1BC2, A2BC1, 1A2BC, and 2A1BC. Taking all of these possibilities together, we get 6 + 2 + 6 + 6 = **20** possible orders.

8. We can find the greatest trime by looking at the prime numbers between 1 and 100, which are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97. We can see from the provided example that in order to form a trime, we can "merge" two double-digit prime numbers that share a digit. This way, we know that at least two of the two-digit numbers produced are prime, and simply need to check the third. Since we're looking for the greatest trime, we can start with 97 and work our way down. 97 can be "merged" with any of the primes in the 70s, so 79, 73, or 71. However, 979 produces 99 as one of its two-digit numbers, 973 produces 93 as one of its two-digit numbers, and 971 produces 91 as one of its two-digit numbers, all of which are not prime. 89 can be "merged" with 97 to make 897, but 87 is not prime. 83 can be merged with 37 or 31, but neither of 87 or 81 are prime. 79 can be "merged" with 97 to make 797, which produces 77 as one of its two-digit numbers, which is not prime. 73 can be "merged" with 37 or 31. While 737 produces 77 as one of its two-digit numbers, which is not prime, 731 produces three two-digit prime numbers: 73, 31, and 71. So, **731** is the greatest possible trime.

2020 School Team Round Solutions

1. There are a total of 93 + 62 + 63 + 127 + 42 = 387 animals at the zoo. With 93 mammals, 93/387 = **24.03**% of the animals in the zoo are mammals, rounded to the nearest hundredth.

2. There are 4! = 4 × 3 × 2 × 1 = 24 total possible arrangements of the four integers 1, 2, 3, and 4. We need to subtract out all of the arrangements that include 2 and 3 in adjacent positions or 1 and 4 in adjacent positions, since 2 + 3 = 5 and 1 + 4 = 5. There are 6 arrangements that include 2 directly followed by 3: 1234, 2314, 2341, 1423, 4123, and 4231. So, we know that there will be an additional 6 arrangements where 3 is directly followed by 2: 1324, 3214, 3241, 1432, 4132, and 4321. We also see that in some of the cases above, we have already accounted for arrangements in which 1 and 4 are adjacent. In particular, we have accounted for arrangements where we see 1 and 4 in adjacent positions at the beginning of the sequence (1423, 1432, 4123, 4132) and at the end of the sequence (2314, 2341, 3214, 3241). So, we need to account for the arrangements where 1 and 4 are in adjacent positions in the middle of the arrangement. These arrangements are: 3412, 3142, 2413, and 2143. So, in total, we have 6 + 6 + 4 = 16 arrangements where we have adjacent numbers that sum to 5. So, there are 24 − 16 = **8** arrangements of 1, 2, 3, and 4 where no two adjacent numbers have a sum of 5.

3. Start by writing an equation to represent what we know about each grade. Let d = dog owners and n = non-dog owners. Since we're asked about the probability of a 7th grader who does not own a dog, we can focus on just the 7th grade. We can represent the 7th grade scenario with $108 = d + n$. We also know that the number of dog owners is twice the number of students who do not own a dog, which can be represented by $d = 2n$. So, we can substitute into the 7th grade equation from above and solve: $108 = 2n + n \rightarrow 108 = 3n \rightarrow n = 36$. In total, there are 93 + 108 = 201 students, so the probability of a seventh grader who does not own a dog winning the raffle is 36/201 = **12/67**.

4. Start with the lower left-hand corner. You know that this cell and the cell directly to the right of it must contain values that sum to 5. Therefore, using the values between 1 and 5, the possibilities to fill these two squares are either 2 and 3 or 1 and 4. However, we can see that there is a 3 already in the lower right-hand corner, in the same row. Since 1 through 5 will appear exactly once in each row, we know that the lower left-hand corner will contain either 1 or 4. We can see there is already a 1 in the left-most column, and since 1 through 5 will appear exactly once in each column, the lower left-hand corner must contain a 4. Continuing to use similar strategies to fill in the cells, we find that the upper left corner will contain a 3 and the upper right corner will contain a 4 (see image). So, the product of the integers in the four corner shaded cells is 3 × 4 × 4 × 3 = **144**.

3	5	2	1	4
1	2	3	4	5
2	3	4	5	1
5	4	1	3	2
4	1	5	2	3

start here

5. The circumference of a circle is represented by $C = \pi d$, where d = diameter. We are given that the circumference of the Earth is 40,075 km. So, we can substitute this value into the formula in order to solve for the diameter of the Earth: $40{,}075 = \pi d \rightarrow 12{,}756.268688815 = d$. We'll need to find the length of the satellite's orbit (i.e. its circumference) in order to figure out how long the satellite's orbit will take. If 12,756.268688815 km is the diameter of the Earth, then in order to find the diameter of the satellite's orbit, we must extend the "line" that is the Earth's diameter to span the satellite's orbit. This means we'll need to add 400 km (the distance between the satellite and Earth) to Earth's diameter on both ends. So, the diameter of the satellite's orbit is 12,756.268688815 + 400 + 400 = 13,556.268688815 km. With this, we can calculate

the circumference of the satellite's orbit. We have $C = \pi(13{,}556.268688815) \rightarrow C = 42{,}588.27412287$ km. Finally, we know that the satellite travels at 28,000 km/h. In order to calculate the time it takes to travel the satellite's orbit at this speed, we need to use the formula for distance, $d = r \times t$, where d = distance, r = rate, and t = time. We can substitute what we know and solve. We have $42{,}588.27412287 = 28{,}000 \times t \rightarrow 1.5210097901 = t$. This means that it takes the satellite 1.5210097901 hours to orbit around the Earth one time. However, we are asked how many minutes this orbit will take. So, knowing there are 60 minutes in 1 hour, take 1.5210097901 × 60 = 91.2605874062, which is 91.3 rounded to the nearest tenth. Therefore, it takes the satellite **91.3** minutes to orbit Earth one time.

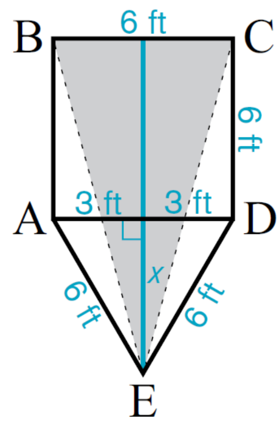
6. There are three possibilities for how the children are divided among the houses. There could be 1 child living in each house; 2 children living in one house and 1 child living in another; or 3 children living in one house. Let's start with the scenario where there is 1 child in each house. There are 3! = 3 × 2 × 1 = 6 possibilities of how the children could be arranged in the houses with 1 child in each house. In 6/6 of these scenarios, the children would play in house B, because no child would have to travel more than one house away from their own. Two children would travel the distance of one house over, and one child would not have to travel at all. If the children were to play in house A or house C, one child would have to travel two houses away from their own while another would have to travel one house away from their own, which is a greater total amount of travel than if they played in house B. Now, let's look at the scenario where 2 children live in one house and 1 child lives in another (meaning there is one house that has no children). Let's say that the three children are named 1, 2, and 3. Within this arrangement, there are three possibilities: children 1 and 2 live together, children 2 and 3 live together, or children 1 and 3 live together. If children 1 and 2 live together, for example, we can still say that we are arranging 3 things: (1,2), 3, and 0 (where 0 represents the empty house). So, there would be 3! = 3 × 2 × 1 = 6 possible arrangements. In two of these arrangements (where children 1 and 2 live together in house B), (i.e. in 2/6 of these arrangements), the children would play in house B. This is because in the two scenarios where children 1 and 2 live in house B, only child 3 would need to travel, and it would only be the distance of one house over. This is the minimum possible distance traveled in this scenario. This situation is the same if children 2 and 3 live together, and if children 1 and 3 live together - there would still be 2/6 arrangements for each case where the children would play in house B. So, there are a total of 6 + 6 + 6 = 18 possibilities of how the children could be arranged in the houses with 2 children in one house and 1 child in the other, and 2 + 2 + 2 = 6 of these arrangements would have the children playing in house B to minimize distance traveled. Finally, there are 3 possible arrangements when all three of the children live in the same house. Either they all live in house A; they all live in house B; or they all live in house C. In only 1 of these scenarios would the children play in house B. This would be when they all live in house B, because they would not have to travel at all to play. Therefore, there are a total of 6 + 18 + 3 = 27 possible arrangements of the children, and 6 + 6 + 1 = 13 of these arrangements would have the children playing in house B. So, the probability of the children playing in house B is **13/27**.

7. If n = ab, and both a and b have values somewhere between 1 and 10, then when a = 1, there are 10 distinct product expressions of a and b (because b could be equal to 10 different values). When a = 2, we would have already multiplied 1 by 2 when a = 1. So, when a = 2, there are 9 distinct product expressions of a and b (because b could be equal to 9 different values, between 2 and 10). Continuing with this reasoning, there will be one fewer distinct product expression of a and b each time the value of a is increased by 1. So, there are 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 55 distinct product expressions of a and b. However, there are some resulting positive integers that we have double counted. For example, while 2 × 8 and 4 × 4 are different product expressions, they result in the same positive integer, 16. So, we'll need to subtract out the product expressions that result in a duplicate positive integer, which can be found below.

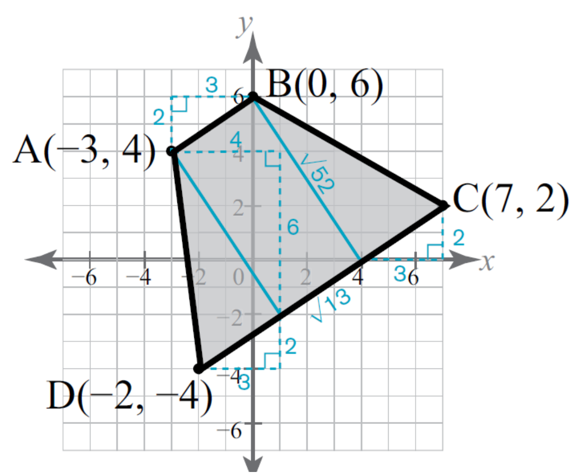
$2 \times 2 = 4 \times 1 = 4$	$2 \times 3 = 6 \times 1 = 6$	$2 \times 4 = 8 \times 1 = 8$	$2 \times 5 = 10 \times 1 = 10$
$2 \times 6 = 3 \times 4 = 12$	$2 \times 8 = 4 \times 4 = 16$	$2 \times 9 = 3 \times 6 = 18$	$2 \times 10 = 4 \times 5 = 20$
$3 \times 3 = 9 \times 1 = 9$	$3 \times 8 = 6 \times 4 = 24$	$3 \times 10 = 5 \times 6 = 30$	$4 \times 9 = 6 \times 6 = 36$
$4 \times 10 = 5 \times 8 = 40$			

You'll notice there are 13 sets above, each of two product expressions that are equal to the same positive integer. We want to "keep" one of each of these positive integers, and simply eliminate the duplicates. So, we find that there are 55 - 13 = **42** distinct positive integers.

8. The formula for the area of a triangle is $A = (1/2)bh$, where b = base and h = height. So, in order to find the area of the shaded triangle, BEC, we'll need to know the measurements for the base and the height of this triangle. Since the base of BEC is shared with the square ABCD, and we know the side lengths of the square are all 6 feet, we know that the base of triangle BEC is 6 feet long. Because we know that ADE is an equilateral triangle, we can say that a line segment from E to the midpoint of segment AD would be perpendicular to AD and would represent the height of triangle ADE. We also know that since AD is also a side of the square, AD is 6 feet in length, as is AE and DE, since ADE is an equilateral triangle. This would also mean that the two smaller triangles formed by drawing the line segment from E to the midpoint of AD would create two halves of AD, each equal to 3 feet in length, which would serve as the bases of the smaller triangles formed. Using this information and the Pythagorean Theorem, we can find the height of triangle ADE. Since $a^2 + b^2 = c^2$, we have $3^2 + b^2 = 6^2 \rightarrow 9 + b^2 = 36 \rightarrow b^2 = 27 \rightarrow b = \sqrt{27}$ feet. If we extend the line segment between E and the midpoint of AD all the way down to BC, we are representing the height of the shaded triangle BEC, since ADE is an equilateral triangle that is exactly aligned with the perfect square ABCD. So, since the square has a height of 6 feet, we can find the height of triangle BEC by adding $6 + \sqrt{27}$. Using this information and the formula for the area of a triangle, we can find the area of BEC. We have $A = (1/2)(6)(6 + \sqrt{27}) \rightarrow A = 3(11.1961524227) \rightarrow A = 33.5884572681$ ft^2. So, the area of triangle BEC is **33.6** ft^2, rounded to the nearest tenth.



9. Start by drawing a line segment from A down to CD that creates a right angle at AB and a right angle at CD. Then, draw another line segment from B down to CD that creates another right angle at AB and another right angle at CD. These lines should divide the original figure into a rectangle and two right triangles. Let's start by looking at the rectangle. Create a right triangle between A and B, where AB is the hypotenuse. You should find that there is one side of this triangle that is of length 2 units and one side that is of length 3 units. Use the Pythagorean Theorem to determine the length of AB. We have $a^2 + b^2 = c^2 \rightarrow 2^2 + 3^2 = c^2 \rightarrow 4 + 9 = c^2 \rightarrow 13 = c^2 \rightarrow \sqrt{13}$ units $= c$. Next, use the same process to find the length of the line segment you've drawn from A to CD (as this is a side of your rectangle). When you create a right triangle where this line segment is the hypotenuse, you should find that you have a triangle with sides of length 4 units and 6 units. Use the Pythagorean Theorem to determine the length of AB. We have $a^2 + b^2 = c^2 \rightarrow 4^2 + 6^2 = c^2 \rightarrow 16 + 36 = c^2 \rightarrow 52 = c^2 \rightarrow \sqrt{52}$ units $= c$. Using the formula for the area of a rectangle, $A = bh$ (where b = base and h = height), calculate the area of the rectangle. We get $A = (\sqrt{13})(\sqrt{52}) \rightarrow A = 26$ units^2. Because the rectangle and each triangle shares a side (of length $\sqrt{52}$ units), you only need to find the length



of the remaining side of each triangle in order to calculate the area of each triangle. Using the same process as described above, you'll find that the missing side length of both triangles is $\sqrt{13}$ units. (The triangles are in fact equal.) Using the formula for area of a triangle, we can calculate the area of each triangle. We have $A = (1/2)bh \rightarrow A = (1/2)(\sqrt{13})(\sqrt{52}) \rightarrow A = 13$ units^2. Therefore, the area of the full figure is 26 + 13 + 13 = **52** units^2.

10. Start by determining the factors of 2020: 1, 2, 4, 5, 10, 20, 101, 202, 404, 505, 1010, 2020. We know that there is no even integer that will share no common factors greater than 1 with 2020 (even integers will always share at least a factor of 2 with 2020). So, divide 1,000,000/2 = 500,000 to get the number of odd integers between 1 and 1,000,000. In doing this, we have ensured that all integers have been eliminated from our pool that would share 2, 4, 10, 20, 202, 404, 1010, and/or 2020 as a factor with 2020. Next, let's look at integers that would share a factor of 5 with 2020. We know that there are 1,000,000/5 = 200,000 integers with 5 as a factor. However, since we have already eliminated all integers with a factor of 10 from our pool with the previous step, we only need to account for integers ending in 5 (which is half of all integers with factors of 5). So, 200,000/2 = 100,000 integers that need to be eliminated from our pool. We're down to 500,000 - 100,000 = 400,000 integers. You'll notice that with these two steps, we have actually already eliminated integers that share any factor with 2020, with the exception of 101. So, let's look at integers that would share 101 as a factor with 2020. When we divide 1,000,000 by 101, we find that we do not get a whole number result. Here, we'll need to round down to the nearest whole number. Since we're looking at only integers between 1 and 1,000,000, we only need to look at the integers in this range with a factor of 101, and there are 9,900 of these integers. Half of these integers will be even (202, 404, 606, etc.), and all even integers have already been eliminated from our pool. So, we can take 9,900/2 = 4,950, which gives us the number of odd integers with a factor of 101. In the previous step, however, we already eliminated all integers from our pool with a factor of 5. This means that certain factors of 101 (505, 1515, etc.) have already been eliminated from our pool. In fact, it will be every 10th multiple of 101 that has already been removed, so 9,900/10 = 990. We can subtract 4,950 − 990 = 3,960 to get the number of integers that share 101 as a factor with 2020 and have not already been eliminated. So, there are 400,000 − 3,960 = **396,040** integers between 1 and 1,000,000 that share no factor greater than 1 with 2020.