MATHCOUNTS®

2020 Chapter Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2020 MATHCOUNTS® Chapter Competition. These solutions provide creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions with the MATHCOUNTS community.

2020 Chapter Competition Sprint Round

1. Each hour has 60 minutes. Therefore, 4.5 hours has 4.5×60 minutes $= 45 \times 6$ minutes $=$ **270** minutes.

2. The ratio of apples to oranges is 5 to 3. Therefore, $\frac{5}{3} = \frac{a}{9}$, where a represents the number of apples. So, $3a = 5 \times 9 = 45$. Thus, $a = \frac{45}{3} = \mathbf{15}$ apples.

3. Substituting for x and y yields $12xy = 12 \times \frac{1}{2} \times 6 = 6 \times 6 = \mathbf{36}$.

4. The minimum Category 4 speed is 130 mi/h. The maximum Category 1 speed is 95 mi/h.
   Absolute difference is $|130 - 95|$ mi/h $=$ **35** mi/h.

5. Area $A = s^2 = 144$ cm^2. Perimeter $p = 4s = 4\sqrt{A} = 4\sqrt{144} = 4 \times 12$ cm $=$ **48** cm.

6. Since 3 miles = 1 league, 1 mile $= \frac{1}{3}$ league $= \frac{1}{3} \times 24$ furlongs $=$ **8** furlongs.

7. The sum of the measures of the 4 interior angles of a quadrilateral is 360 degrees. Therefore, $360 = 119 + 89 + 49 + m\angle D = 257 + m\angle D$, so $m\angle D = 360 - 257 = \mathbf{103}$ degrees.

8. In any nonzero geometric sequence, the ratio of the nth term to the $(n - 1)$th term, for $n > 1$, is constant, and is called the common ratio. Therefore, $t_5/t_4 = t_2/t_1 = 4/2 = 2$, so $t_5 = 2t_4 = 2 \times 16 = \mathbf{32}$.

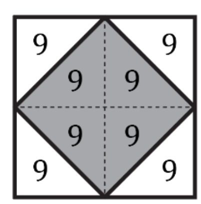
9. The first polygon, a triangle, has 3 sides. The other polygon has $2 \times 3 - 2 = 6 - 2 = \mathbf{4}$ sides.

10. Solve using cross cancelation to get $50 \text{ cm} \times \frac{\$4.00}{1 \text{ meter}} = 50 \text{ cm} \times \frac{1 \text{ meter}}{100 \text{ cm}} \times \frac{\$4.00}{\text{meter}} = \frac{1}{2} \times \$4.00 = \$\mathbf{2.00}$ *or* $\$\mathbf{2}$.

11. The average of 70 over 3 rounds is the sum of the scores for the 3 rounds divided by 3. Therefore, the sum of the scores over 3 rounds is $3 \times 70 = 210$. The two known scores are 68 and 72, which sum to 140. Therefore, the third round score must have been $210 - 140 = \mathbf{70}$.

12. The diagonals of the inner shaded square are congruent to the sides of the outer square—call that length s. Then the area of the inner shaded square is $s^2/2$ while the area of the outer square is s^2, which is 2 times as big as the inner square. Because the inner square has area 36 cm^2, the area of the outer square is 2×36 cm^2 $=$ **72** cm^2.

    Alternatively, if we draw the diagonals of the shaded square, we create 8 congruent 45-45-45 right triangles. Since the shaded square, which is made up of four of these right triangles, has area 36 cm^2, it follows that each triangle has area $36/4 = 9$ cm^2. Therefore, the outer square, composed of 8 of these right triangles, has area $9 \times 8 = \mathbf{72}$ cm^2.

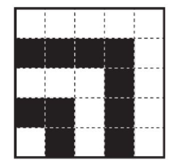


13. Because Rafa averages moving 12.7 meters per point and Sascha averages 11.8 meters per point, Rafa averages $12.7 - 11.8 = 0.9$ meters per point more than Sascha. Given 320 points for the match at that rate, Rafa will run a total of $320 \times 0.9 = \mathbf{288}$ meters more than Sascha.

14. Let l be the length and w be the width of the rectangle. Then $l + w = 16$ u and $l = 3w$. Thus, we have 16 units $= 3w + w = 4w$, $w = 4$ units, $l = 16$ units $- 4$ units $= 12$ units, and $lw = 12$ units $\times$ 4 units = **48** units^2.

15. The expression $\sqrt{5 \times 6 \times 10 \times 12}$ can be rewritten as $\sqrt{5 \times 10 \times 6 \times 12}$. Simplifying, we get $5\sqrt{2} \times 6\sqrt{2} = 30 \times 2 =$ **60**.

16. The smallest, unshaded square has area $(1 \text{ in})^2 = 1^2 \text{ in}^2 = 1 \text{ in}^2$. The area formed by the smallest, unshaded square combined with the smallest shaded strip is $(2 \text{ in})^2 = 4 \text{ in}^2$. Thus, the area of the smallest shaded strip is $4 \text{ in}^2 - 1 \text{ in}^2 = 3 \text{ in}^2$. The area of the smallest, unshaded square plus the smaller, shaded strip plus the next unshaded strip is $(3 \text{ in})^2 = 9 \text{ in}^2$. Adding the next shaded strip yields $(4 \text{ in})^2 = 16 \text{ in}^2$. The absolute difference of the last two yields the area of the larger shaded strip: $16 \text{ in}^2 - 9 \text{ in}^2 = 7 \text{ in}^2$. The total area of the two shaded strips is $3 \text{ in}^2 + 7 \text{ in}^2 =$ **10** in^2.



Alternatively, if we divide the figure, as shown, into 25 squares, each 1 inch × 1 inch, we can see that 10 of these squares are shaded, giving us a total area of $10 \times 1 \text{ in}^2 =$ **10** in^2.

17. The positive two-digit integers with 1 as units digit are 11, 21, 31, 41, 51, 61, 71, 81, 91, of which $21 = 3 \times 7$, $51 = 3 \times 17$, $81 = 9 \times 9$ and $91 = 7 \times 13$ are not prime, leaving **5** numbers that are.

18. Let l be the number of laps Mira walked. Equating the two donations yields: $\$(10 + 0.10l) = \$0.35l$. Simplifying, we get $10 = 0.35l - 0.10l = 0.25l = l/4$. Therefore, $l = 4 \times 10 =$ **40** laps.

19. Let m represent the first addend 4A☐, with temporarily allowing each of A and ☐ to represent any digit 0 through 9 meaning $400 \le m \le 499$. Let n represent the second addend 53☐, with temporarily allowing ☐ to represent any digit 0 through 9 meaning $530 \le n \le 539$. We have an additional criterion that the sum must be of the form 1☐☐☐, so letting each ☐ represent, independently, any digit 0 through 9 means we must have $1000 \le m + n \le 1999$. For A = 9, $499 + 539 = 1038$, so this case works. For A = 8, $489 + 539 = 1028$, so this case works. For A = 7, $479 + 539 = 1018$, so this case works. For A = 6, $469 + 539 = 1008$, so this case works. For A = 5, $459 + 539 = 998$, so this case does not work. In fact any smaller A (less than or equal to 5) requires n to be at least $1000 - 459 = 541$ just to reach the bare minimum allowed sum, but this is above its maximum allowed value for n, so A can have any one of the **4** values 6, 7, 8, or 9.

20. The left end (0 inches) is directly under the $2\frac{1}{2}$ in mark. Therefore, the reading on the lower ruler will always be $2\frac{1}{2}$ in less than the reading on the upper ruler, wherever the two rulers overlap, so when the upper ruler shows 6 in, the lower ruler will show 6 inches $- 2\frac{1}{2}$ inches $= 3\frac{1}{2}$ inches. So, $q =$ $\mathbf{3\frac{1}{2}}$.

21. An integer is divisible by 6 if and only if it is divisible by 2 and by 3. An integer is divisible by 2 if and only if its units digit is 0, 2, 4, 6 or 8, and the number in question has units digit 2, so it is divisible by 2. An integer is divisible by 3 if and only if the sum of its digits is divisible by 3. The sum of the three known digits is $5 + 1 + 2 = 8$, which is 1 short of being a multiple of 3. Therefore, the two undetermined digits must add to a multiple of 3 plus 1; because digits cannot exceed 9, the sum of the two undetermined digits cannot exceed 18. As a result, the two digits in question can have of sum of 1, 4, 7, 10, 13 or 16. The desire is for the maximum value, which is 16 for the sum. A 9 in the left blank and a 7 in the right blank yields the greatest result, **59,172**.

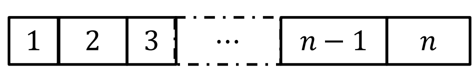
22. Recall that $3^2 = 9$; $3^4 = 3^2 \times 3^2 = 9 \times 9 = 81$; $3^6 = 9 \times 81 = 729$, which gets us close to 1500. So, now multiply by 3 rather than 3^2 to get $3^7 = 3 \times 729 = 2187$. Because $x < y$ if and only if $3^x < 3^y$, having $729 = 3^6 < 1500 = 3^x < 2189 = 3^7$ means that $6 < x < 7$. To know whether x is closer to 6 or to 7, we need to know about $3^{6.5} = 3^6 \times 3^{0.5} = 729\sqrt{3} < 729 \times 2 = 1458 < 1500$, meaning that $6.5 < x < 7$, so the integer that x is closest to is **7**.

23. There are 9 points in the grid. Two distinct points determine a line. There are $\frac{9\times 8}{2} = 36$ combinations of pairs of points. However, there is a bit of a problem in that when we have 3 distinct points on some lines, meaning there are $\frac{3\times 2}{2} = 3$ pairings of points that generate the same line, so we are counting those lines 3 times instead of just once. Take the top row of dots for example: The same line is generated by the left and middle dots, the left and right dots, and the middle and right dots—a total of 3, just as the calculation said we would find. How many sets of 3 collinear dots do we have? There are 3 such horizontal rows, 3 such vertical rows, and 2 such corner-to-opposite-corner diagonals, for a total of 3 + 3 + 2 = 8. There is no other way to overcount lines due to having 3 or more collinear points in our figure. Thus, we have counted 8 of the lines 3 times each, for a total of $8 \times 3 = 24$ times, instead of just 8 times, for an overcount of 16. Therefore, the number of distinct lines is $36 - 16 = \mathbf{20}$ inches.

24. We have $a_3 + a_2 + a_1 + a_0 = a_3 \cdot 1^3 + a_2 \cdot 1^3 + a_1 \cdot 1 + a_0 = (2 \times 1 - 3)^3 = (-1)^3 = \mathbf{-1}$.

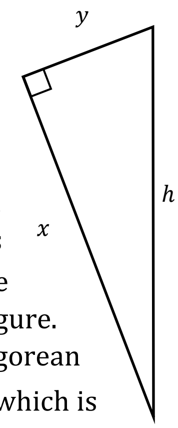
25. The number of marbles that take one path versus the other at a fork is proportional to ratio of the cross-sectional areas of the pipes for each path, which, in turn, is proportional to the squares of the given diameters. At the top fork, the left to right ratio of marbles is $10^2 : 20^2$ = 100:400 = 1:4. As fractions, the left path gets $\frac{1}{1+4} = \frac{1}{5}$ of the marbles and the right path gets $\frac{4}{1+4} = \frac{4}{5}$ of the marbles. Since there are 10,000 marbles, 10,000/4 = 2000 go left and the remaining 8000 go right at the top fork. The bottom row has three pipes: one on the left (L), one in the middle (M), and one on the right (R)—the only one we really care about is M. The fork between L and M gets 2000 marbles, which are split proportionally $30^2 = 900$ for L to $10^2 = 100$ for M. Thus, $\frac{100}{900+100} = \frac{1}{10}$ of the 2000, or 200, go through M. The fork between M and R gets 8000 marbles, which are split proportionally $10^2 = 100$ for M to $20^2 = 400$ for R. Thus, $\frac{100}{100+400} = \frac{1}{5}$ of the 8000, or 1600, go through M. Therefore, the total going through M is $200 + 1600 = \mathbf{1800}$ marbles.

26. Let's start with one row of n rectangles all the same height (width is unimportant) lined up end to end:



There are n individual rectangles, $n - 1$ pairs of adjacent rectangles [1-2, 2-3, 3-4, …, $(n-2)$-$(n-1)$, $(n-1)$-n]. We have $n - 2$ groupings of 3 consecutive individual rectangles [1-2-3, 2-3-4, …, $(n-2)$-$(n-1)$-n]. This pattern continues until we have 2 groupings of $n - 1$ consecutive individual rectangles, and ultimately a count of 1 for the full rectangle containing all n individual rectangles, for a total of $n + (n-1) + (n-2) + \cdots + 2 + 1 = \frac{n(n+1)}{2}$ distinct rectangles. If we have m copies of such rows, these form, independently, $\frac{m(m+1)}{2}$ options for rows of rectangles, and $\frac{l(l+1)}{2}$ options for l layers. Our 3 × 4 × 5 prism produces a total of $\left[\frac{3\times 4}{2}\right]\left[\frac{4\times 5}{2}\right]\left[\frac{5\times 6}{2}\right] = 6 \times 10 \times 15 = \mathbf{900}$ prisms.

27. There are 10 seats, of which 6 are occupied. We care only about each seat being occupied versus unoccupied, not who is occupying it, so order does not matter. We need the number of combinations of 10 things taken 6 at a time: $\frac{10\times9\times8\times7}{4\times3\times2\times1} = 210$ seating arrangements. These can be categorized into 3 layout patterns: (1) 3 fully occupied benches and 2 unoccupied benches; (2) 2 fully occupied benches, 2 partially occupied benches and 1 unoccupied bench; (3) 1 fully occupied bench, 4 partially occupied benches. We are interested in the probability for case (1) or case (2), which is 1 minus the probability of case (3). For case (3), any one of the 5 benches is equally likely to be fully occupied, so we get 5 equally likely options. For each of those options, there are 4 partially occupied benches, each with 2 options as to which of the 2 seats is occupied. Altogether we have $5 \times 2^4 = 80$ arrangements with no empty bench out of 210 total seating arrangements. Therefore, our desired probability is $1 - \frac{80}{210} = \mathbf{\frac{13}{21}}$.



28. The area enclosed by the square is the sum of the three given areas, or 2 m^2 + 2 m^2 + 2 m^2 = 6 m^2. Since the square's area is its side length squared, it follows that each side is $\sqrt{6}$ meters. The long sides of the kite are given as length x; let's call the length of the short sides y. The vertical diagonal of the kite is congruent to each side of the square, thus $\sqrt{6}$ meters. Each of the two long sides and adjacent short side form the legs of a right triangle whose hypotenuse is the vertical diagonal of the kite. One of the triangles in shown in the figure. The area of the triangle is $xy/2 = 1\text{ m}^2$, so $xy = 2\text{ m}^2$ and $y = 2\text{ m}^2/x$. Based on the Pythagorean Theorem, $x^2 = h^2 - y^2 = 6 - \frac{4}{x^2}$. Multiplying through by x^2, we get $(x^2)^2 - 6x^2 + 4 = 0$, which is an equation that is quadratic in x^2. Let's use the quadratic formula to solve for x^2, which is our quantity of interest: $x^2 = \frac{6 \pm\sqrt{(-6)^2-4\times4}}{2} = 3 \pm \frac{\sqrt{20}}{2} = \left(3 \pm \sqrt{5}\right)\text{ m}^2$. We must be careful about the $\pm$. The $+$ gives the square of the longer side, x^2; the – gives the square of the shorter side, which is actually y^2. Therefore, $x^2 = \left(3+\sqrt{5}\right)\text{ m}^2$, which is of the form $\left(a+\sqrt{b}\right)\text{ m}^2$, with $a = 3, b = 5$, so $a + b = \mathbf{8}$.

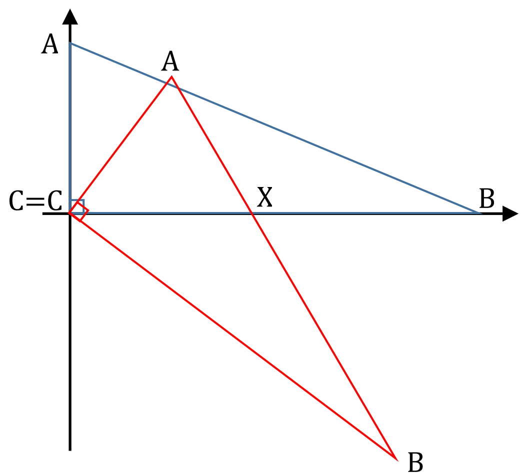
29. The number of 0s that an integer ends in when expressed in base 9 is the number of factors of $9 = 10_9$ the integer has and the rightmost nonzero digit in base 9 is the remainder upon dividing by 9 the integer with all of the factors of 9 removed.

$9! = 9\times8\times7\times6\times5\times4\times3\times2\times1 = 9\times3\times6\times8\times7\times5\times4\times2$ [move 3 and 6; drop 1]

$= 9\times9\times16\times7\times5\times4\times2 = 9^2\times16\times280.$ [combine 3s to make 9s, merge all 9s together and do the trivial parts of the remaining multiplications].

The exponent on the base 9 is $m = 2$. The product 16×280 has the same remainder when divided by 9 as does the product of the remainders of each of 16 and 280 when divided by 9. Also an integer and the sum of the digits of that integer have the same remainder when divided by 9. Thus, 16 and $1 + 6 = 7$ have the same remainder, 7 when divided by 9, while 280, $2 + 8 + 0 = 10$, and $1 + 0 = 1$ have the same remainder when divided by 9. Therefore, 16×280 and 7×1 have the same remainder, 7, when divided by 9 (much easier than multiplying 16×280 and then dividing by 9). Thus, $n = 7$, and $(m, n) = (\mathbf{2}, \mathbf{7})$.

30. Given congruent right triangles ABC and A′B′C′, A = (0, 5), B = (12, 0), C = (0, 0), A′ = (3, 4), C′ = (0, 0), and X is the intersection of $\overline{A'B'}$ and the x-axis. What is the area enclosed by triangle XB′C′ divided by the area of A′B′C′? The area enclosed by A′B′C′ is the same as that of ABC, which is $12 \times 5/2 = 30$ units². The slope of C′A′ is 4/3, so the slope of perpendicular C′B′ is the negative reciprocal, or −3/4. The length of $\overline{C'B'}$ is the same as the length of CB, namely 12 units. Therefore, the coordinates of B′ are $B'_x = 0 + \frac{4}{\sqrt{3^2+4^2}} \times 12 = +9.6$ and $B'_y = 0 - \frac{3}{\sqrt{3^2+4^2}} \times 12 = -7.2$, so B′ = (9.6, −7.2). Based on the y-component values, X is $\frac{4}{4+7.2} = \frac{4}{11.2} = \frac{5}{14}$ of the way from A′ to B′, so $X_x = 3 + \frac{5}{14}(9.6 - 3) = \frac{75}{14}$. Therefore, the base of XB′C′ is the length of $\overline{C'X}$, which is $\frac{75}{14}$ while the height is $7.2 = \frac{36}{5}$, making the enclosed area equal to one-half the product of these two values: $\frac{1}{2} \times \frac{75}{14} \times \frac{36}{5} = \frac{270}{14} = \frac{135}{7}$, which needs to divided by 30, the area enclosed by A′B′C′: $\frac{135}{30\times7} = \mathbf{\frac{9}{14}}$.



2020 Chapter Target Round Solutions

1. Natasha has 14 – 8 = 6 more candy bars than Soren. To be equal, Natasha must give Soren 1/2 of her "excess" 6, thus **3** candy bars.

2. A vertical line along $x = 7$ splits the figure in question into two triangles: to the left of the line is a triangle of base 6 units and height 3 units for an area of 9 units, and to the right of the line is a triangle of base of 5 units and height 8 units for an area of 20 units^2. The two triangles combined have area $9 + 20 = \mathbf{29}$ units^2.

3. The factor theorem says polynomial $P(x)$ has $x - r$ as a factor if and only if $P(r) = 0$, so $5x - 3 = 5\left(x - \frac{3}{5}\right)$ is a factor if the polynomial has value 0 at $x = 3/5$: $0 = 5\left(\frac{3}{5}\right)^2 + 7\left(\frac{3}{5}\right) + k = \frac{9}{5} + \frac{21}{5} + k = 6 + k$, so $k = \mathbf{-6}$. Alternatively, since $5x - 3$ is a factor of the polynomial, then $(5x - 3)(x + a) = 5x^2 + 7x + k$. Expanding the left side of the equation yields $5x^2 + 5ax - 3x - 3a = 5x^2 + (5a - 3)x - 3a$. That must mean that $5a - 3 = 7$ and $5a = 10$ so $a = 2$. That also means that $k = -3a = -3(2) = \mathbf{-6}$.

4. Dividing, we get $\frac{\$35.3\cancel{\times 10^6}}{4.29\cancel{\times 10^6}} = \$8.228 \approx \$\mathbf{8.23}$. [NOTE: The given values are clearly rounded off and could represent values of total gross being between \$35.25 million and \$35.35 million and total tickets being between 4.285 million and 4.295 million, for which the most extreme values would yield an average ticket price between \$8.21 and \$8.25, so in reality, the result may be as much as \$0.02 off from what we calculated, but the best estimate that we can give with the data provided is \$8.23, and only that value.]

5. Let x and y be two distinct nonzero numbers. We have $x + y = xy = x/y$. $xy = x/y$ implies $xy^2 = x$, so $y^2 = 1$, meaning $y = +1$ or $y = -1$. Now, $y = 1$ requires $x + 1 = x$, which is nonsense, so we must have $y = -1$. Then $x - 1 = -x$, so $x = \frac{1}{2}$ and $-y = \frac{1}{2} + 1 = \mathbf{\frac{3}{2}}$.

6. The lengths of the legs of the triangle are in the ratio (shorter to longer) $\frac{6.2\ \cancel{\text{cm}}}{9.3\ \cancel{\text{cm}}} = \frac{2}{3}$, so the ratio of the lengths of short leg to long leg to hypotenuse is $2 : 3 : \sqrt{13}$. Area is proportional to square of length, so the ratio of the three corresponding areas is $4 : 9 : 13$. Therefore, the area of the smallest profile is 4/13 the area of the largest profile, thus $\frac{4}{13} \times 25\ \text{cm}^2 = \mathbf{\frac{100}{13}}\ \text{cm}^2$.

7. The original figure shown in the problem presents a dodecagon with the 6 diagonals that connect opposite vertices, all intersecting at the dodecagon center. The dodecagon and the 6 shown diagonals partition the dodecagon and its interior into 12 isosceles triangles. All 12 isosceles triangles have legs whose length is half that of the given diagonals, namely 5 meters. Each side of the dodecagon constitutes the base of one of the isosceles triangles—8 of them have length $\sqrt{10}$ meters and 4 have length $\sqrt{2}$ meters, so we need to find the area enclosed by those triangles—half the base times the height. We are given the base, and the hypotenuse is the 5 meters. We use the Pythagorean Theorem to find the base length of each triangle. For the triangles with base length $\sqrt{10}$ meters, the height is $\sqrt{5^2 - \left(\frac{\sqrt{10}}{2}\right)^2} = \sqrt{25 - 2.5} = \sqrt{22.5}$ meters. That means there are 8 triangles with area $\frac{1}{2}\sqrt{10} \times \sqrt{22.5} = \frac{1}{2}\sqrt{225} = \frac{15}{2}$ m^2. For the triangles with base length $\sqrt{2}$ meters, the height is $\sqrt{5^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{25 - 0.5} = \sqrt{24.5}$ meters. That means there are 4 triangles with area $\frac{1}{2}\sqrt{2} \times \sqrt{24.5} = \frac{1}{2}\sqrt{49} = \frac{7}{2}$ m^2. The total area of the dodecagon, then, is $8 \times \frac{15}{2} + 4 \times \frac{7}{2} = 60 + 14 = \mathbf{74}$ m^2.

8. To be ahead, Team South can have a score of:
   1, with Team North at 0, for probability $0.27 \times 0.55 = 0.1485$;
   2, with Team North at 0 or 1, for probability $0.05 \times (0.55 + 0.33) = 0.0440$;
   3, with Team North at 0 or 1 or 2, for probability $0.01 \times (0.55 + 0.33 + 0.10) = 0.0098$.

   The total probability is $0.1485 + 0.0440 + 0.0098 = 0.2023 = 20.23\ \%$, which rounds to **20** %.

2020 Chapter Team Round Solutions

1. Jack spent 10.44 gal × \$2.47/gal = \$25.79 in Las Cruces, NM. He spent 10.01 gal × \$2.41/gal = \$24.12 in Tucson, AZ. He spent 10.39 gal × \$2.65/gal = \$27.53 in Flagstaff, AZ. And he spent 9.16 gal × \$2.62/gal = \$24.00 in Las Vegas, NV. Jack spent the greatest amount in Flagstaff, AZ \$**27.53**.

2. Start with the layout of 5 rows and 5 columns of cells with only the two shaded cells filled in with a number. Start with the shaded cell containing 23 (because it is the smaller value). Fill in any empty adjacent (horizontally or vertically, not diagonally) cell with 24 (1 more than 23). We don't use 22, because that doesn't move us closer to a number that is 1 more or 1 less than the 26 that is already filled in. Furthermore, we are trying to maximize the values in these cells. So, next we put 25 in any empty cell adjacent to a cell containing 24. Then put 26 in any empty cells adjacent to a cell containing 25. Then put 27 in any empty cell adjacent to a cell containing 26 (including the shaded cell). Then put 28 in any empty cell adjacent to a cell containing 27. Finally, put 29 in the last empty cell. Once every cell is filled, we see that the greatest value is **29**.

23	24	25	26	27
24	25	26	27	28
25	26	27	28	**29**
26	27	28	27	28
27	28	27	26	27

3. Marko slept from 10:00 p.m. to 5:00 a.m., a total of 7 hours. At a rate of 65 times per minutes, which equals 3900 times per hour, his heart beat 7 × 3900 = 27,300 times. He exercised from 5:30 a.m. to 7:30 a.m., a total of 2 hours. At a rate of 118 times per minute, which equals 7080 times per hour, his heart beat 2 × 7080 = 14,160 times. During the remaining 24 – 9 = 15 hours, Marko had normal activity. At a rate of 72 times per minute, which equals 4320 times per hour, his heart beat 15 × 4320 = 64,800 times. So, his heart beat a total of 27,300 + 14,160 + 64,800 = $\mathbf{106,260}$ times.

4. Let t represent the orbit time and d represent the maximum distance from the Sun. We are given that t^2 is proportional to d^3, so t is proportional to $\sqrt{d^3}$. Therefore, $t_2/t_1 = \sqrt{(d_2/d_1)^3}$, or more conveniently, $t_2 = t_1\sqrt{(d_2/d_1)^3}$, where the subscripts 1 and 2 refer to the two planets being compared. In this case, planet 2 is Neptune; for planet 1 we are given information for three planets when any one by itself will suffice. If we use Mars for planet 1, we get: $t_2 = 1.882 \text{ yr} \times \sqrt{(4.50 \times 10^9/(2.28 \times 10^8))^3} = 1.882 \times \sqrt{19.74^3} =$ 165.02 years, which we would round to **165** years. In fact, the values for all 3 given planets result in 164.94 years to 165.02 years—all rounding to the same **165** years.

5. When there are n variables that all take on nonnegative integer values that sum to s, the number of distinct ordered n-tuple solutions is given by the number of combinations of $n + s - 1$ things taken s at a time, that is $\frac{(n+s-1)!}{(n-1)!\,s!}$. In this problem, $s = 22$, and there are 3 variables, so n = 3. Therefore, we have $\frac{24!}{2!\,22!} = \frac{24\times23}{2\times1} =$ 276 equally likely ordered triples as solutions. However, some of these involve ordered triples for which 2 of the components are equal, so there is no order of the components with $a < b < c$. The two that are equal can be anywhere from 0 to 11 inclusive, allowing the third component to be nonnegative. It can be any one of the 3 components that is the unequal one making 12 × 3 = 36 "problem cases", leaving 240 cases with 3 mutually distinct values. Of these, 1 out of every 6, for a total of 40, is in the right order, for a probability of $\frac{40}{276} = \mathbf{\frac{10}{69}}$.

   Alternatively, the following tables provide an organized listing of the 52 combinations of three nonnegative integers a, b and c for which $a + b + c = 22$. As indicated, there are 40 ordered triples for which $a < b < c$.

0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11
22	21	20	19	18	17	16	15	14	13	12	11

4	4	4	4	4	4
4	5	6	7	8	9
14	13	12	11	10	9

1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10
20	19	18	17	16	15	14	13	12	11

5	5	5	5
5	6	7	8
12	11	10	9

2	2	2	2	2	2	2	2	2
2	3	4	5	6	7	8	9	10
18	17	16	15	14	13	12	11	10

6	6	6
6	7	8
10	9	8

3	3	3	3	3	3	3
3	4	5	6	7	8	9
16	15	14	13	12	11	10

7
7
8

To determine the total number of ordered triples (a, b, c) for which $a + b + c = 22$, we can count the number of arrangements for each of these 52 combinations and add them together. Each of the 40 combinations with three distinct integers has $3! = 6$ arrangements, and each of the 12 combinations in which two of the three integers are the same has $\frac{3!}{2} = 3$ arrangements. That's $6(40) + 3(12) = 240 + 36 = 276$ arrangements in total. Therefore, the desired probability is $\frac{40}{276} = \mathbf{\frac{10}{69}}$.

6. Let's regard a clockwise rotation as negative and a counterclockwise rotation as positive. Then on each roll, one of +1, −2, +3, −4, +5, or −6 steps will be taken. In order to get back to the starting point, the sum of the three counts of steps must be a multiple of 6. This means either 0 or 2 of the rolls must be odd. (Otherwise, the sum is odd and cannot be a multiple of 6.) The patterns of working 3-roll combinations are as follows: For 3 even rolls, the combinations are −2, −2, −2, with 1 order; −2, −4, −6, with 6 orders; −4, −4, −4, with 1 order; and −6, −6, −6, with 1 order. For 2 odd rolls, the combinations are 1, 1, −2, with 3 orders; 1, 3, −4, with 6 orders; 1, 5, −6, with 6 orders; 3, 3, −6, with 3 orders; 3, 5, −2, with 6 orders; and 5, 5, −4, with 3 orders. That gives us a total of 1(3) + 6(4) + 3(3) = **36** sequences.

7. What sorts of numbers must be abundant or must not be abundant? Powers of prime numbers cannot be abundant: To have $1 + p + \cdots + p^{n-1} + p^n > 2p^n$ requires $1 + p + \cdots + p^{n-1} > p^n$, but the left side is $\frac{p^n - 1}{p - 1}$, with numerator less than p^n and denominator at least 1. This knocks out 1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, 37, 41, 43, 47, 49, 53, and 59. Semiprime numbers that are the product of two distinct prime numbers cannot be abundant: To have $1 + p + q + pq > 2pq$ requires $1 + p + q > pq$, but the left side is at most $2\max(p, q) \le pq$. This knocks out 6, 10, 14, 15, 21, 22, 26, 33, 34, 35, 38, 39, 46, 51, 55, 57, and 58. Perfect numbers are numbers the sum of whose factors is exactly twice the number, in which case an integer $k > 1$ multiplied by the prime number is abundant. The first two perfect numbers are 6 and 28, making 12, 18, 24, 30, 36, 42, 48, 54, and 56 abundant. This leaves 20, 40, 44, 45, 50, and 52 to evaluate individually. [Note: In number theory $\sigma(n)$ or $\sigma_1(n)$ is often used to indicate the sum of the divisors of n.]

$\sigma(20) = \sigma(2^2 \times 5^1) = (1 + 2^1 + 2^2)(1 + 5^1) = 7 \times 6 = 42 > 2 \times 20$, so abundant.

40, as a multiple of an abundant number, is also abundant.

$\sigma(44) = \sigma(2^2 \times 11^1) = (1 + 2^1 + 2^2)(1 + 11^1) = 7 \times 12 = 84 \le 2 \times 44$, so not abundant.

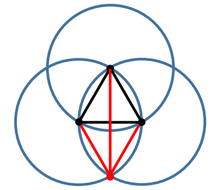
$\sigma(45) = \sigma(3^2 \times 5^1) = (1 + 3^1 + 3^2)(1 + 5^1) = 13 \times 6 = 78 \le 2 \times 45$, so not abundant.

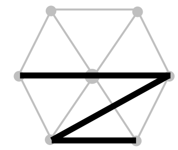
$\sigma(50) = \sigma(2^1 \times 5^2) = (1 + 2^1)(1 + 5^1 + 5^2) = 3 \times 31 = 93 \le 2 \times 50$, so not abundant.

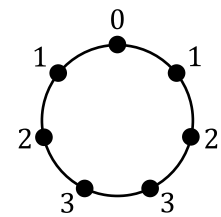
$\sigma(52) = \sigma(2^2 \times 13^1) = (1 + 2^1 + 2^2)(1 + 13^1) = 7 \times 14 = 98 \le 2 \times 52$, so not abundant.

The sum of these abundant numbers is $12 + 18 + 20 + 24 + 30 + 36 + 40 + 42 + 48 + 54 + 56 = \mathbf{380}$.

8. Let's start with 3 points. There is 1, and only 1, way for the 3 resulting line segments to be congruent: when the points form the vertices of an equilateral triangle as shown. Is there a way to add a 4th distinct point such that the 3 newly formed segments are all congruent to the first 3 segments? Each of the three congruent smaller circles shows the points that are the same distance from the point that is the center of the circle as are the other two points. In order for all the 6 segments to be the same length, the new point must be placed where all 3 circles intersect—but there is no such point. The best we can do is put the 4th point where two of the circles intersect. These 4 points form a rhombus whose 4 sides and short diagonal are congruent, but the long diagonal is not, for two lengths of line segments. It might appear that stacking of equilateral triangles would always yields the minimum number of distinct lengths of segments for n points. The case here, $n = 7$, does work; black line segments in the figure to the right indicate the 3 distinct sizes of segments formed by 6 vertex points of a regular hexagon and its center point. However, this technique works for only a few values of n, and 7 just happens to be one of them. A fully general technique, working for all n, involves organizing the n points as the vertices of a regular n-gon (n-sided polygon) or, equivalently, equally spaced around a circle, as shown. Label any one of the points as #0. Due to rotational symmetry of order n, it does not matter which point is so picked—all the points have identical behavior. Label each vertex adjacent to #0 as #1. Continue around the circle labeling each vertex 1 more than its predecessor until all of them are numbered. If n is even, there will be one vertex across the diameter from #0 that is labeled $n/2$; otherwise (as in our case of $n = 7$), there will be two adjacent vertices labeled $(n-1)/2$ with their midpoint being across the diameter from #0. In either case, the highest-numbered point[s] will be $\lfloor n/2 \rfloor$, where the notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and is often called the floor of x. The shortest line segments have the length between #0 and #1; the second shortest line segments have the length between #0 and #2; this pattern continues until the longest line segments have the length between #0 and #$\lfloor n/2 \rfloor$. Again, due to symmetry, it does not matter whether one follows the vertices around the left versus around the right. In summary, there are $\lfloor n/2 \rfloor$ distinct lengths of line segments, each in the form of the length of the line segment connecting #0 and #m for $1 \le m \le \lfloor n/2 \rfloor$, in strictly increasing order. [Note that the length of the segment connecting #1 and #3 is the same as that between #0 and #2, ...] Therefore, the answer to this problem is $\lfloor 7/2 \rfloor = \mathbf{3}$ lengths.







9. Any card can be chosen first and that establishes the suit—there is no such thing as a wrong suit for the first card. However, each card after that must be of the same suit as the first. For the second pick there are 12 cards left of the same suit as the first out of 51 cards left to pick. If the second pick matches, then there remain 11 of the same suit out of 50 total left to pick third. If all picks have matched so far, there remain 10 of the same suit out of 49 total to pick fourth. If still all picks have matched, there remain 9 of the same suit out of 48 total to pick fifth. These probabilities are independent of previous picks, thus multiplicative, so the overall probability is: $\frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48} = \frac{12}{48} \times \frac{11}{49} \times \frac{10}{50} \times \frac{9}{51} = \frac{1 \times 11 \times 1 \times 3}{4 \times 49 \times 5 \times 17} = \frac{33}{16\,660} = 0.001\,98\ldots$, which rounds to the nearest thousandth as **0.002**.

10. Any right triangle will fit interior to a circle if the hypotenuse of the triangles is less that the diameter of the circle. A right triangle whose hypotenuse has the same length as a diameter of the circle can be inscribed in the circle with the hypotenuse coinciding with the diameter. Therefore, to fail this requirement with the given data requires the hypotenuse to exceed the diameter, which is twice the radius, or $2\sqrt{100} = 20$. Thus, we must find a Pythagorean triple whose hypotenuse exceeds 20. The least such hypotenuse is 25, which can occur with 7-24-25 and with 15-20-25. To meet the given criteria (minimize the area), the best situation is to spread the lengths of the two legs as far apart as possible, so the 7-24-25 will win, with an area of $7 \times 24/2 = \mathbf{84}$ units^2