

MATHCOUNTS®

2019 Chapter Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2019 MATHCOUNTS® Chapter Competition. These solutions provide creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

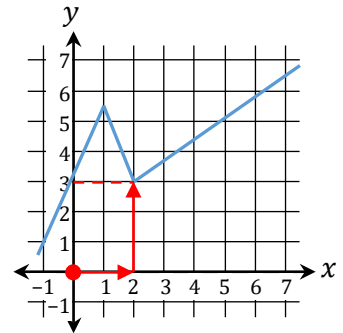
We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

*Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions
with the MATHCOUNTS community.*

2019 Chapter Sprint Round Solutions

1. For a nonnegative real number n , the square of the square root of n yields n . Since the square root of n is 4, n must be the square of 4, which is $4 \times 4 = \mathbf{16}$.

2. We start at the origin, where the two double-arrowed axis lines meet, and, since we want $x = 2$, we proceed horizontally to the right (toward the x) to where we see the 2. We then proceed vertically parallel to the axis marked y until we meet the blue function line, which we see to be 3 units above the x -axis. So the answer is **3**.



3. Line up the values vertically so that the units digit of all the values are in a column as in the figure with the problem and add the digits column by column:

$$\begin{array}{r}
 1 \\
 10 \\
 101 \\
 1\ 010 \\
 10\ 101 \\
 \underline{101\ 010} \\
 \mathbf{112\ 233}
 \end{array}$$

4. $8^2 - 6^2 = 64 - 36 = \mathbf{28}$.

5. With 63 people owning a dog, the remaining $100 - 63 = 37$ people must own just a cat, since all 100 people own a dog, a cat or both. However, 58 people own a cat, and only 37 cats have been accounted for. That means $58 - 37 = \mathbf{21}$ cats are owned by the people owning a dog also.

6. $6 \frac{\text{apples}}{\text{bin}} \times 4 \frac{\text{bins}}{\text{bundle}} \times 2 \frac{\text{bundles}}{\text{crate}} = \mathbf{48} \frac{\text{apples}}{\text{crate}}$.

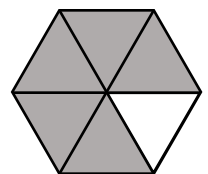
7. If all 9 coins were pennies, then they would have a value of 9¢. Because the total value is to be 29¢, we are 20¢ short. Each time we replace a penny by a nickel, the total value increases by 4¢. To make up that 20¢ by substituting nickels for pennies will require $\frac{20¢}{4¢} = 5$ such substitutions, thus ending up with **5** nickels.

8. There are 60 seconds in a minute, so $\frac{32 \text{ heartbeats}}{15 \text{ sec}} \times \frac{60 \text{ sec}}{1 \text{ min}} = \frac{32 \times 60 \text{ heartbeats}}{15 \text{ min}} = 32 \times 4 \frac{\text{heartbeats}}{\text{min}} = \mathbf{128} \frac{\text{heartbeats}}{\text{min}}$.

9. The first five prime numbers in increasing order are 2, 3, 5, 7, 11. The median value is the third prime, **5**.

10. $90\% = 9/10$, so $108 = \frac{9}{10}x$ and $x = \frac{10}{9} \times 108 = 10 \times \frac{108}{9} = 10 \times 12 = \mathbf{120}$.

11. A regular hexagon is composed of 6 congruent equilateral triangles as shown in the figure. The shaded region covers 5 of the 6 congruent triangles, so the shaded area is **5/6** of the hexagon's area.



12. $2 = \frac{2n}{8} = \frac{n}{4}$ so $n = 4 \times 2 = \mathbf{8}$.

13. Here we have three smaller triangles joined in a row such that each pair of adjacent triangles forms a triangle, with there being two such pairs (left with middle and middle with right) and all three taken together forming a single triangle. That means $3 + 2 + 1 = 6$ total triangles. Had n distinct line segments been drawn from the vertex to the base of the original triangle, partitioning the original triangle into $n + 1$ small triangles, the total number of triangles would have been the triangular number $T_{n+1} = (n + 1)(n + 2)/2$.
14. In the Fastball row under Min Speed we see the value 80 mi/h. In the Knuckleball row under Max Speed we see the value 70 mi/h. The absolute difference is $|80 - 70|$ mi/h = $|10|$ mi/h = **10** mi/h.
15. The first six positive prime numbers are 2, 3, 5, 7, 11 and 13. We do not need to determine the actual product, only how many zeros are at the right end of the integer—in other words how many factors of 10 are generated by the product. The only ways to get factors of 10 are by multiplying by a multiple of 10 and by multiplying by a multiple of 2 and by a multiple of 5. In the first six primes there are no multiples of 10, but there is one multiple of 2 and one multiple of 5, having one factor of 2 and one factor of 5, respectively, which combine to yield one factor of 10, so the number of 0s at the right end of the product is **1**.
16. $\frac{10 \text{ lap}}{10 \text{ min}} = 1 \frac{\text{lap}}{\text{min}} \times 2.5 \frac{\text{mi}}{\text{lap}} \times 60 \frac{\text{min}}{\text{h}} = 1 \times 2.5 \times 60 \frac{\text{mi}}{\text{hr}} = \mathbf{150} \frac{\text{mi}}{\text{h}}$.
17. The sum of the two solutions of the quadratic equation $Ax^2 + Bx + C = 0$ is given by $-B/A$ [and, as a side note, the product of the solutions is C/A]. Because the two solutions are given as 2 and 7, the sum of the two solutions is 9. Now, $A = 1$ and $B = a$, so $9 = -\frac{B}{A} = -\frac{a}{1} = -a$, so that $a = \mathbf{-9}$.
18. The way of selecting the most total days without selecting three consecutive days by starting at either end and alternately selecting two and skipping one. Thus, starting from the left, days 1, 2, 4, 5, and 7 are selected without three in a row anywhere. There are only two days left to choose, #3 and #6. Choosing either one makes at least three consecutive days. Thus, making 5 selections is not enough to guarantee three consecutive days but **6** is enough.
19. $(3 \oplus 4) \oplus (20 \ominus 16) = \sqrt{3^2 + 4^2} \oplus \sqrt{20^2 - 16^2} = 5 \oplus 12 = \sqrt{5^2 + 12^2} = \mathbf{13}$.
 The evaluations of the radicals can be done in any of several ways:
 Just crunch numbers: $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$; $\sqrt{20^2 - 16^2} = \sqrt{400 - 256} = \sqrt{144} = 12$;
 $\sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$.
 Recognize each square root except the last is re-squared, so $\sqrt{3^2 + 4^2 + 20^2 - 16^2} = \sqrt{9 + 16 + 400 - 256} = \sqrt{169} = 13$.
 Note Pythagorean triples: 3-4-5 scaled by 1 and 4 to 3-4-5 and 12-16-20, respectively; and lastly 5-12-13.
 The \oplus operation combines two legs to yield the hypotenuse.
 The \ominus operation combines the hypotenuse and one leg to yield the other leg.
20. Jones catches up 800 m in 4 min and, thus at a rate of: $\frac{800 \text{ m}}{4 \text{ min}} = 200 \frac{\text{m}}{\text{min}} \times \frac{1 \text{ km}}{1000 \text{ m}} \times \frac{60 \text{ min}}{1 \text{ h}} = 12 \frac{\text{km}}{\text{h}}$. Jones is going 50 km/h, so the car is going $50 - 12$ km/h = **38** km/h.

21. Each of 20 people on the winning team shook hands with each of 20 people on the losing team for a total of $n = 20 \times 20 = 400$ handshakes. Each of the 20 people on the winning team fist-bumped each of the other 19 teammates, but we are double counting because player A fist-bumping player B is the same action as player B fist-bumping A. So, there were $m = 20 \times 19/2 = 10 \times 19 = 190$ fist bumps.
 $n + m = 400 + 190 = \mathbf{590}$.

22. The sections of spinner 1 are the prime numbers less than 10: 2, 3, 5, 7. The sections of spinner 2 are the positive perfect squares less than 40: 1, 4, 9, 16, 25, 36.

The 2 on spinner 1 is not relatively prime to these values on spinner 2: 4, 16, and 36.

The 3 on spinner 1 is not relatively prime to these values on spinner 2: 9 and 36.

The 5 on spinner 1 is not relatively prime to these values on spinner 2: 25.

The 7 on spinner 1 is not relatively prime to these values on spinner 2: —.

There are, thus, 6 pairs that qualify; there are $4 \times 6 = 24$ total pairs.

The desired probability is, therefore, $6/24 = \mathbf{1/4}$.

23. The sum of the three values is $(20 + A) + (30 + A) + (40 + A) = 90 + 3A$, but we are told that their sum is $100 + A$. For $90 + 3A = 100 + A$, we must have $2A = 10$, so $A = \mathbf{5}$.

24. There are $4 \times 4 \times 4 = 64$ equally likely outcomes of rolling the die. The only ways to get a sum of 7 are:

rolling 4, 2 and 1 in any order, which can occur in $\frac{3!}{1! 1! 1!} = \frac{6}{1 \times 1 \times 1} = 6$ distinct, equally likely ways;

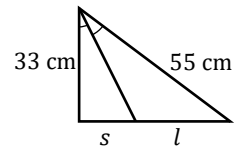
rolling 3,3 and 1 in any order, which can occur in $\frac{3!}{2! 1!} = \frac{6}{2 \times 1} = 3$ distinct, equally likely ways;

rolling 3, 2 and 2 in any order, which can occur in $\frac{3!}{1! 2!} = \frac{6}{1 \times 2} = 3$ distinct, equally likely ways.

Thus, the probability that the sum of the numbers rolled is 7 is $\frac{6+3+3}{64} = \frac{12}{64} = \frac{\mathbf{3}}{\mathbf{16}}$.

25. We are to determine s , given the information in the figure shown and $s + l = 44$ cm.

The reason for bisecting the angle formed by the 33 cm and 55 cm sides is that we are instructed to bisect the maximal acute angle. Now, a 33 cm-44 cm-55 cm triangle is a 3-4-5 right triangle with a scaling of 11 cm, so the angle opposite the 55 cm side is a right angle, not acute. The larger acute angle is then the angle opposite the next longest



side, which is the 44 cm side. We can use the triangle angle-bisector theorem, which says that $\frac{l}{s} = \frac{55 \text{ cm}}{33 \text{ cm}}$, so

$\frac{l+s}{s} = \frac{55 \text{ cm} + 33 \text{ cm}}{33 \text{ cm}} = \frac{88}{33} = \frac{8}{3}$. Because $l + s = 44$ cm, we have $\frac{44 \text{ cm}}{s} = \frac{8}{3}$, so $s = \frac{3 \times 44 \text{ cm}}{8} = \frac{3 \times 11}{2} \text{ cm} = \frac{\mathbf{33}}{\mathbf{2}} \text{ cm}$.

26. The only Q is in the center square, so that is where we must start. From there, moving left, right, up, or down, gets us to a U, so that is 4 ways to make QU. Due to symmetry all four ways behave alike, so we can analyze just one choice for the U and multiply by 4. Regardless of which U we move to, three out of the four possible moves left, right, up, or down get us to an E to make QUE. We must be careful here, though, because not all 3 Es yield the same number of options for the next move:

(i) We see that 2 of the Es are interior on a diagonal, each of which leads to any of 4 Us. From any of these Us, we can move to any of 3 Es to make QUEUE.

(ii) But 1 of the Es is along an outer edge in the middle and leads to any of 3 Us. From any of these Us, we can move to any of 3 Es to make QUEUE.

Thus, we have a total of $4[(2 \times 4 \times 3) + (1 \times 3 \times 3)] = 4(24 + 9) = 4(33) = \mathbf{132}$ ways.

27. Here we have a problem relating the average speed over a whole trip to the average speed over each of several legs of equal length. The overall average is the harmonic mean of the speeds for each leg. For two legs with average speeds of v_1 and v_2 , respectively, the harmonic mean is $\frac{1}{\frac{1}{2}(\frac{1}{v_1} + \frac{1}{v_2})} = \frac{2v_1v_2}{v_1+v_2}$. Thus, we have the overall average speed being $v_a = \frac{2v_1v_2}{v_1+v_2}$. We are given the actual value for the difference between v_1 and v_2 , but for now let's simply say that $v_2 = v_1 + d$. For solving equations with lots of peculiar numbers and measurement units, it is often easier to solve in terms of general variables and substitute actual quantities once we express the variable to be solved in terms of the other variables. We have $v_a = \frac{2v_1v_2}{v_1+v_2}$, and $v_a = \frac{2v_1(v_1+d)}{2v_1+d}$, but we need to rearrange this to express v_1 in terms of v_a and d . Multiplying both sides by $2v_1 + d$ yields $v_a(2v_1 + d) = 2v_1(v_1 + d)$, which expands to $2v_av_1 + dv_a = 2v_1^2 + 2dv_1$. Rearranging this into conventional form for a quadratic equation in terms of v_1 yields $2v_1^2 + 2(d - v_a)v_1 - dv_a = 0$. Using the quadratic formula, we get $v_1 = \frac{-2(d-v_a) \pm \sqrt{[2(d-v_a)]^2 - 4(2)(-dv_a)}}{2 \times 2} = \frac{(v_a-d) \pm \sqrt{(v_a-d)^2 + 2dv_a}}{2}$. We know that the overall average speed is given by the total distance $2 \times 180 \text{ mi} = 360 \text{ mi}$ divided by the total time 7.5 h. So, we have $v_a = \frac{360 \text{ mi}}{7.5 \text{ h}} = \frac{3 \times 120 \text{ mi}}{3 \times 2.5 \text{ h}} = \frac{4 \times 120 \text{ mi}}{4 \times 2.5 \text{ h}} = \frac{480 \text{ mi}}{10 \text{ h}} = 48 \frac{\text{mi}}{\text{h}}$. We are given that $v_1 = v_2 - 20 \frac{\text{mi}}{\text{h}}$, so $d = 20 \frac{\text{mi}}{\text{h}}$ and $v_a - d = 48 \frac{\text{mi}}{\text{h}} - 20 \frac{\text{mi}}{\text{h}} = 28 \frac{\text{mi}}{\text{h}}$. Substituting into our quadratic formula yields:

$$v_1 = \frac{28 \frac{\text{mi}}{\text{h}} \pm \sqrt{(28 \frac{\text{mi}}{\text{h}})^2 + 2 \times 20 \frac{\text{mi}}{\text{h}} \times 48 \frac{\text{mi}}{\text{h}}}}{2} = 14 \frac{\text{mi}}{\text{h}} \pm \sqrt{196 \left(\frac{\text{mi}}{\text{h}}\right)^2 + 480 \left(\frac{\text{mi}}{\text{h}}\right)^2} = 14 \frac{\text{mi}}{\text{h}} \pm \sqrt{676} \frac{\text{mi}}{\text{h}} = (14 \pm 26) \frac{\text{mi}}{\text{h}}$$

This yields a choice of 40 mi/h or -12 mi/h, but a negative result does not make sense for this problem, so the answer must be $v_1 = 40 \text{ mi/h}$.

28. Start with 20 and work up, incorporating new factors that have not yet been covered. Now, we are dealing with numbers in the 20s (or less due to common factors)—let's just say an average of $25 = 10^2/2^2$. With 7 such factors, we would be at most about $(10^2/2^2)^7 = 10^{14}/2^{14} \approx \frac{10^{14}}{1.6 \times 10^4} \approx 6 \times 10^9$, so let's work 20 through 27 and see where we stand. [NOTE that *billion* refers to 10^9 .]

20	$2^2 \times 5$;
21 = 3 × 7, both new, so 20 × 21	$2^2 \times 3 \times 5 \times 7$;
22 = 2 × 11, with new 22 but old 2, so 20 × 21 × 11	$2^2 \times 3 \times 5 \times 7 \times 11$;
23 is a new prime, so 20 × 21 × 11 × 23	$2^2 \times 3 \times 5 \times 7 \times 11 \times 23$;
24 = 2 ³ × 3 with one new 2, two old 2's and old 3, so 20 × 21 × 22 × 23	$2^3 \times 3 \times 5 \times 7 \times 11 \times 23$;
25 = 5 ² with a new and an old 5, so 100 × 21 × 22 × 23	$2^3 \times 3 \times 5^2 \times 7 \times 11 \times 23$;
26 = 2 × 13 with a new 13 and an old 2, so 100 × 21 × 22 × 23 × 13	$2^3 \times 3 \times 5^2 \times 7 \times 11 \times 13 \times 23$;
27 = 3 ³ with two new and one old 3, so 100 × 21 × 22 × 23 × 117	$2^3 \times 3^3 \times 5^2 \times 7 \times 11 \times 13 \times 23$;

Pause: Let's assess where we are with $100 \times 21 \times 22 \times 23 \times 117$. The product of the first 4 factors is straightforward to do exactly with only mental power; for the rest an approximation is likely good enough. $21 \times 22 \times 23 = 22 \times (22^2 - 1^2) = 22^3 - 22 = 2^3 \times 11^3 - 22 = 8 \times 1331 - 22 = 10\,648 - 22 = 10\,626$. The 100 part tacks on two more 0's for 1 062 600, just over 10^6 ; 117 is only a little over 10^2 , for a total of a little over 10^8 . We need another factor of 10 or so to get there.

28 = 2² × 7, all of which is old, so nothing new to contribute;

29 is a prime, so new, and provides us with our needed factor of at least 10, making the answer **29**.

29. My favorite approach to this type of problem is somewhat unconventional—but only in the sense of combining properties of similarity and proportionality in such a way as to combine steps unconventionally, but quite validly, to reduce the count and complexity of steps, but all equivalent to a more traditional approach. The area of a triangle is $1/2$ times the base times the height. For the ratio of areas of two triangles the $1/2$ cancels out, leaving the product of the ratio of the bases time the ratio of the heights. Let's regard the base of $\triangle ABC$ to be segment BC and the base of $\triangle PQR$ to be segment PQ . It is very convenient they are parallel to one another. With $\frac{AP}{AB} = \frac{AQ}{AC} = \frac{1}{5}$, $\frac{PQ}{BC}$ must be the same $\frac{1}{5}$, so we know the ratio of the bases is $\frac{1}{5}$. Now we need to deal with the ratio of the heights of the two triangles, and this will be a 2-step process: first, the ratio of the height of PQ above BC to the height of A above BC , and, second, the ratio of the height of PQ above R to the height of PQ above BC . The product of these two will be the ratio of the height of PQ above R to the height of A above BC , which product is the ratio of the heights of the two triangles of concern. We use similarity to realize that the height of A above PQ is $\frac{1}{5}$ the height of A above BC , so the ratio of the height of PQ above BC to the height of A above BC is 1 minus the ratio $\frac{1}{5}$, which is $\frac{4}{5}$ as the answer to the first part. For the second part, we need to determine the horizontal distance from line BQ to line CP going horizontally at some height, and left-to-right will be regarded as a positive value and right-to-left as a negative value (a form of what is often referred to a displacement [physics] or directed distance [mathematics]). At the height of base PQ , we are going from Q to P , which is right-to-left, so negative. Since we care only about ratios and scaling, we can regard the distance as normalized to -1 . At the base BC , the point on BQ is B and the point on CP is C , with C to the right of B , so a positive distance that is 5 times the magnitude of the distance -1 from Q to P , with our given scaling. The magnitude of -1 is 1 , which multiplied by 5 is 5 . The two lines intersect at R so the distance from R to R is 0 . Now, 0 is $\frac{1}{6}$ of the way from -1 to 5 , so the height of R below PQ is $\frac{1}{6}$ the height of BC below PQ , which is $\frac{4}{5}$ the height of BC below A , making the height of $\triangle PQR$ equal to $\frac{4}{5} \times \frac{1}{6} = \frac{2}{15}$ the height of $\triangle ABC$. With a ratio of bases being $\frac{1}{5}$ and the ratio of heights being $\frac{2}{15}$, the ratio of the areas is $\frac{1}{5} \times \frac{2}{15} = \frac{2}{75}$.

[The explanation is long because there are some atypical details to discuss, but the calculations are short when you are used to the concept. It is very quick to see that the base of $\triangle RPQ$ is $1/5$ as wide as for $\triangle ABC$, and then the height of $\triangle RPQ$ is $\frac{1}{6} \times \frac{4}{5} = \frac{2}{15}$ that of $\triangle ABC$, so the ratio of the areas is $\frac{1}{5} \times \frac{2}{15} = \frac{2}{75}$.]

30. We can reorder the terms so that $a \leq b \leq c$. It must be that a must be 2 or 3 ; otherwise, a being 1 makes the sum of the three reciprocals greater than 1 , and the maximum sum for $a > 3$ is $\frac{3}{4} < \frac{6}{7}$. For the sum to have a denominator of 7 , at least one of a , b , and c must be a multiple of 7 , and it is not a since a is 2 or 3 ; neither can it be b since then c would also have to be at least 7 , and the sum of the reciprocals cannot reach the requisite $\frac{6}{7}$. Thus, with $a = 3$ and $c \geq 7$, then $\frac{1}{b} \geq \frac{6}{7} - \left(\frac{1}{3} + \frac{1}{c}\right) = \frac{8}{21}$, so $b \leq \frac{21}{8} < 3$, making $b < a$ (a contradiction of specified conditions) if $a = 3$. Thus, a must be 2 , making $\frac{1}{b} + \frac{1}{c} = \frac{6}{7} - \frac{1}{2} = \frac{5}{14}$. Therefore, $\frac{14}{5} < b \leq \frac{14}{3}$ (the first condition based on c being an arbitrarily large integer, making $1/c$ arbitrarily close to 0 , and the second condition based on $c \geq 7$ since c is a multiple of 7). Thus, b must be 3 or 4 . For $b = 3$, c would have to be $c = \frac{1}{\left(\frac{5}{14} - \frac{1}{3}\right)} = \frac{1}{\frac{1}{42}}$, which is the reciprocal of an integer and, therefore, suitable. For completeness (but not necessary in the competition), for $b = 4$, c would have to be $c = \frac{1}{\left(\frac{5}{14} - \frac{1}{4}\right)} = \frac{3}{28}$, which is not the reciprocal of an integer, so is not an acceptable answer. Therefore, we end up with: $a = 2, b = 3, c = 42; a + b + c = 2 + 3 + 42 = 47$.

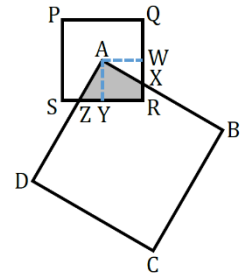
2019 Chapter Target Round Solutions

1. The price increase of each orange is $\$0.69 - \$0.49 = \$0.20$. Therefore, the price increase of 6 oranges is $6 \times \$0.20 = \mathbf{\$1.20}$. [WARNING: That second decimal place is required for monetary amounts.]
2. The y -intercept is determined by solving for y when $x = 0$. For Chris that is at $y = 7$, while for Sebastian that is $y = b$. For Sebastian's to be double Chris', $b = 2 \times 7 = 14$.
The x -intercept is determined by solving for x when $y = 0$. For Chris that is at $x = -7/3$ while for Sebastian that is $x = -b/a$. For Sebastian's to be double Chris', $-\frac{b}{a} = -\frac{14}{a} = 2\left(-\frac{7}{3}\right) = -\frac{14}{3}$, so $\frac{1}{a} = 1/3$ and $a = 3$.
Therefore, $a + b = 3 + 14 = \mathbf{17}$.

3. In 12 min, the Cubes have gained $21 - 18 = 3$ points over the Bisectors, so the rate of gain is $\frac{3 \text{ points}}{12 \text{ min}} = \frac{1 \text{ point}}{4 \text{ min}}$, or 1 point every 4 minutes. The game lasts $2 \times 20 \text{ min} = 40 \text{ min}$. Therefore, the total gain of points by the Cubes over the Bisectors is $\frac{1 \text{ pt}}{4 \text{ min}} \times 40 \text{ min} = \mathbf{10}$ points.

4. Let point X be the point of intersection of the bisector of $\angle D$ and line AB , forming $\triangle ADX$, and point Z be the point of intersection of the two dashed lines, forming $\triangle BXZ$. We are given $m\angle XAD = 84^\circ$ and $m\angle ADX = 32^\circ/2 = 16^\circ$, so $m\angle DXA = 180^\circ - (84^\circ + 16^\circ) = 80^\circ$. $\angle DXB$ is the supplement of $\angle DXA$, and, thus, has a measure of 100° ; this is the same as $\angle BXZ$. We are given $m\angle ZDB = 32^\circ/2 = 16^\circ$. Therefore, for $\angle XZB$, which is the angle in question, $m\angle XZB = 180^\circ - (100^\circ + 16^\circ) = \mathbf{64^\circ}$.

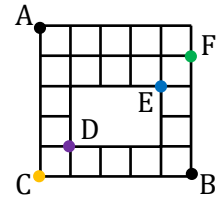
5. As the figure shows, we can draw line segments \overline{AW} and \overline{AY} perpendicular to lines QR and RS , respectively. Because A is the center of square $PQRS$, W and Y are the midpoints of their respective sides of the square. Triangles AWX and AYZ are congruent. The shaded area is quadrilateral $AXRZ$. That area is equal to the sum of the areas of quadrilateral $AXRY$ and triangle AYZ . Since triangles AWX and AYZ are congruent, their areas are equal. Thus, the given shaded area is equal to the sum of the areas of quadrilateral $AXRY$ and triangle AWX , which is the same as the area of square $AWRY$ —a square that is similar to square $PQRS$, with linear scale factor of $\frac{1}{2}$. The ratio



of the area $AWRY$ to the area of $PQRS$ is the square of this linear scale factor, thus $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Note that this answer does not depend in any way on the degree of rotation of the larger square about the center of the smaller square.

6. Let the desired tens digit be d , the wrongly input tens digit be w , and the units digit be u . Then the desired number to square is $10d + u$ and the wrongly squared number is $10w + u$. The difference in their squares is $2340 = (10w + u)^2 - (10d + u)^2 = (100w^2 + 20wu + u^2) - (100d^2 + 20du + u^2) = 100(w^2 - d^2) + 20(w - d)u$. Dividing both sides by 20 yields: $117 = 5(w^2 - d^2) + (w - d)u = [5(w + d) + u](w - d)$. We are dealing with ten's digits d and w of 2-digit numbers, with $w > d$, so we must have $1 \leq d < w \leq 9$. Thus, $1 \leq w - d \leq 8$ and $w - d$ must divide $117 = 3^2 \times 13$, meaning that $w - d$ is 1 or 3. If $w - d = 1$, then $5(w + d) + u = 117$, but $w \leq 9$, $d \leq 8$, and $u \leq 9$, so $5(w + d) + u \leq 94$ and cannot be 117. Therefore, $w - d = 3$. Now we have $117 = [5(w + d) + u](3)$, so $5(w + d) + u = 39$, which means: $w + d = 6$ and $u = 9$, or $w + d = 7$ and $u = 4$. Since $w - d$ is odd (3), $w + d$ must likewise be odd. Thus, $w + d = 7$ and $u = 4$ holds. Now, we are to find $(10w + u) + (10d + u) = 10(w + d) + 2u = 10 \times 7 + 2 \times 4 = \mathbf{78}$.

7. In order to traverse from point A to point B, one must pass through exactly one of points C, D, E or F shown. There is some number of paths from A to each of the points C, D, E and F, and for each such path there is some number of paths from there to B. So, we can determine the product of those two values for each point C, D, E and F and add the four products together. The number of paths between any two lattice points that is r steps right and d steps down is the number of combinations of $r + d$ things taken d at a time.



- (i) For $A \rightarrow C$: $r = 0, d = 5$ and ${}_5C_5 = 1$ and for $C \rightarrow B$: $r = 5, d = 0$ and ${}_5C_0 = 1$.
So, it follows that there is $1 \times 1 = \underline{1}$ path from $A \rightarrow C \rightarrow B$.
- (ii) For $A \rightarrow D$: $r = 1, d = 4$ and ${}_5C_4 = 5$ and for $D \rightarrow B$: $r = 4, d = 1$ and ${}_5C_1 = 5$.
So, it follows that there are $5 \times 5 = \underline{25}$ paths from $A \rightarrow D \rightarrow B$.
- (iii) For $A \rightarrow E$: $r = 4, d = 2$ and ${}_6C_2 = 15$ and for $E \rightarrow B$: $r = 1, d = 3$ and ${}_4C_3 = 4$.
So, it follows that there are $15 \times 4 = \underline{60}$ paths from $A \rightarrow E \rightarrow B$.
- (iv) For $A \rightarrow F$: $r = 5, d = 1$ and ${}_6C_1 = 6$ and for $F \rightarrow B$: $r = 0, d = 4$ and ${}_4C_4 = 1$.
So, it follows that there are $6 \times 1 = \underline{6}$ paths from $A \rightarrow F \rightarrow B$.

Total number of paths is $1 + 25 + 60 + 6 = \mathbf{92}$ paths.

8. For the product of integers to be 1, each integer must be +1 or -1, and the count of -1 factors must be even. This means that for a factor that is a difference of two letter values, it is necessary (but not sufficient) that the letters be consecutive in terms of assignment to the sequential numbers 1, 2, 3, 4, 5, 6. Therefore, M and A must be consecutive, so either M, A (yields $M - A = -1$) or A, M (yields $M - A = +1$); T must be consecutive with both H and E , so T must be between H and E , as E, T, H or H, T, E ; E must be consecutive with both L and T , so E must be between L and T , as L, E, T or T, E, L . Putting the last two lines together yields L, E, T, H (yields $(T - H)(L - E)(T - E) = (-1)(-1)(+1) = +1$) or H, T, E, L (yields $(T - H)(L - E)(T - E) = (+1)(+1)(-1) = -1$). Thus, we need M, A to go with H, T, E, L , either order, and A, M to go with L, E, T, H , either order.

Thus, we have 4 choices that satisfy the product criterion:

$$M = 1, A = 2, H = 3, T = 4, E = 5, L = 6;$$

$$H = 1, T = 2, E = 3, L = 4, M = 5, A = 6;$$

$$A = 1, M = 2, L = 3, E = 4, T = 5, H = 6;$$

$$L = 1, E = 2, T = 3, H = 4, A = 5, M = 6.$$

There are $6! = 720$ possible arrangements of the letters with the numeric values. Therefore, the fraction with the desired result is $\frac{4}{720} = \frac{1}{180}$.

2019 Chapter Team Round Solutions

1. Let's try a greedy algorithm, adding one triangle at a time, maximizing the contribution to the perimeter each time. If we attach a new triangle onto a side of 5 inches, that 5-inch side has disappeared, but we now have 14 inches showing, for a net increase of $14 - 5 = 9$ inches to the perimeter. If we attach a new triangle onto a side of 7 inches, that 7-inch side has disappeared, but we now have 12 inches showing, for a net increase of $12 - 7 = 5$ inches to the perimeter. Thus, we should want to attach along a side of 5 inches whenever possible and attach along a side of 7 inches only if such is the only choice. First, we start with one triangle and have perimeter 2×7 inches + 5 inches = 19 inches, and we have two 7-inch and one 5-inch sides to attach to. Let's attach the second triangle along the one available 5-inch side, increasing the perimeter to 19 inches + 9 inches = 28 inches, and we have four 7-inch sides to attach to. Since we have only 7-inch sides to attach the third triangle to, doing so increases the perimeter to 28 inches + 5 inches = **33** inches.

2. $5^{50} = 5^{2 \times 25} = (5^2)^{25} = 25^{25}$, so $n = \mathbf{25}$.

3. Let's use L for the amount Lior has at any given time, C for the amount Celine has at any given time, and E for the amount Elliot has at any given time.

We start off with $L = \frac{2}{3}$, $C = 0$, $E = 0$.

C takes $1/4$ of L : $C = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$, $L = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}$, $E = 0$.

E takes $1/2$ of L : $E = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, $L = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, $C = \frac{1}{6}$.

C and E take [add to their own] $1/3$ of L , thus $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$, which is also the amount L keeps: $C = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$, with E ending up with $\frac{1}{3}$ and L with $\frac{1}{12}$, though there is no need to determine the last two.

4. Being the region between two concentric circles, an annulus has an area equal to the difference of the area within the larger circle minus the area within the smaller circle, thus of the form $\pi(R^2 - r^2)$. The radius R of the larger circle is easy to determine—the distance between the center point of the rotation and the farthest vertex. (Remember, we really need the square of the distance, so there is no need to do the square root portion of the Pythagorean formula only to square it again later.)

Between $(0; -3)$ and $(2; 0)$: $(2 - 0)^2 + [0 - (-3)]^2 = 4 + 9 = 13$.

Between $(0; -3)$ and $(10; 2)$: $(10 - 0)^2 + [2 - (-3)]^2 = 100 + 25 = 125$.

Between $(0; -3)$ and $(6; 6)$: $(6 - 0)^2 + [6 - (-3)]^2 = 36 + 81 = 117$.

Therefore, $R^2 = 125$.

The radius of the inner circle is not so easy—it is not necessarily the distance between the center point of rotation and the nearest vertex, as the shortest distance could be on a side between two vertices. Drawing a quick sketch is likely to convince under time pressure of a competition that the nearest point on the triangle is a vertex, and that is good enough. If you are really nervous and paranoid, though, the only time issues can arise (and it might not be serious even then) is when the negative reciprocal of the slope of a side is between the value of the slope between the rotation center point and each of the two end vertices of that side. This does occur for the side between $(10; 2)$ and $(6; 6)$, inclusive, for which the minimum distance squared between the rotation center point and that side is 112.5, which is greater than the distance squared between the rotation center point and vertex $(2; 0)$, so we will take 13 as the minimum distance squared, r^2 . Therefore, the answer is $\pi(R^2 - r^2) = \pi(125 - 13) = \mathbf{112\pi}$ units².

5. To average the same overall speed s , Crystal must spend $\frac{1.0-0.7}{1.1-0.7} = \frac{0.3}{0.4} = \frac{3}{4}$ of her time at speed 1.1s and $\frac{1.1-1.0}{1.1-0.7} = \frac{0.1}{0.4} = \frac{1}{4}$ at speed 0.7s, which means $\frac{11}{10} \times \frac{3}{4} = \frac{33}{40}$ of the distance at speed 1.1s.

6. If we number the rows of the table as r , with $r = 1$ for the top content (non-header) row to $r = 6$ for the bottom row, and the columns as c , with $c = 1$ for the left content (non-header) column and $c = 6$ for the right column, then the value in the cell at row r and column c is $6(r - 1) + c$, with the first part depending on only the row number and the last part depending on only the column number. The header column at the far left tells how many cells in that row are used and, therefore, how many times $6(r - 1)$ shows up in the sum due to row r ; similarly, the header row at the top tells how many cells in that column are used and, therefore, how many times c shows up in the sum due to column c . By splitting this way, each part of the value expression is handled exactly once and is in a meaningful way—no double counting, etc. Therefore, the sum of the coin-occupied values is:

$$3 \times 0 + 4 \times 6 + 3 \times 12 + 2 \times 18 + 4 \times 24 + 2 \times 30 + 2 \times 1 + 2 \times 2 + 1 \times 3 + 5 \times 4 + 3 \times 5 + 5 \times 6 = 0 + 24 + 36 + 36 + 96 + 60 + 2 + 4 + 3 + 20 + 15 + 30 = \mathbf{326}.$$

7. For a product of factors in the range of 1 to 10 to have a units digit of 0 (be divisible by 10), either 10 is one of the factors, or else 5 is one of the factors along with at least one even number. We must be carefully in assessing the probability because both conditions may occur together. Let's look at two main cases:

1. Of the 4 values, 1 of them is a 10; we do not care what the other 3 values are.

2. Of the 4 values, 1 of them is a 5, at least 1 (meaning 1, 2, or 3) is an even number other than 10 (2, 4, 6, 8), and the remaining ones (2, 1, or 0) is an odd number other than 5.

The two cases were deliberately chosen to be mutually exclusive, so the probabilities of the two can be merely added. We can use the generalized or multivariate hypergeometric probability distribution.

For case 1, we have $\frac{\binom{1}{1}\binom{9}{3}}{\binom{10}{4}} = \frac{1 \times 84}{210} = \frac{84}{210}$. (We will wait until the end to reduce the fractions.)

For case 2, we can decompose into 3 mutually exclusive subcases: 1, 2, or 3 values chosen from 2, 4, 6, and

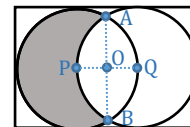
8, yielding $\frac{\binom{1}{1}\binom{1}{0}\binom{4}{1}\binom{4}{2}}{\binom{10}{4}} + \frac{\binom{1}{1}\binom{1}{0}\binom{4}{2}\binom{4}{1}}{\binom{10}{4}} + \frac{\binom{1}{1}\binom{1}{0}\binom{4}{3}\binom{4}{0}}{\binom{10}{4}} = \frac{\binom{1}{1}\binom{1}{0}}{\binom{10}{4}} \left[\binom{4}{1}\binom{4}{2} + \binom{4}{2}\binom{4}{1} + \binom{4}{3}\binom{4}{0} \right] =$

$$\frac{1}{210} (4 \times 6 + 6 \times 4 + 4 \times 1) = \frac{52}{210}.$$

Therefore, the total probability is $\frac{84}{210} + \frac{52}{210} = \frac{136}{210} = \frac{68}{105}$.

8. The rectangle height, 12, is the diameter of each circle, so the radius of the circles is 6.

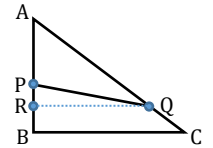
Let's put the origin of a coordinate system at the center O of the rectangle with the x -axis horizontal and y -axis vertical. Then the centers of circles Q and P are at $(\pm 3; 0)$ and the two circles intersect at A and B , $(0; \pm 3\sqrt{3})$. The area of the shaded region is the area of



circle P minus the area of overlap between the two circles. The overlap region can be partitioned into a segment of circle P bounded by \overline{AB} and arc BQA , and a segment of circle Q bounded by \overline{AB} and arc BPA ; these two segments are congruent, so the area of the overlap region is double the area of either segment. We can find the area of segment $AOBQA$ by subtracting the area of the sector of circle P formed by arc BQA minus the area of $\triangle PAB$. $\angle AOP$ and $\angle BOP$ are \perp . The ratio of each of legs \overline{AO} and \overline{BO} to leg \overline{OP} is $\sqrt{3}$, meaning that $m\angle APO = m\angle BPO = 60^\circ$, so the measure of $\angle APB$ and arc BQA are $2 \times 60^\circ = 120^\circ$. Thus, the sector of circle P formed by arc BQA is $1/3$ of circle P and has area 12π ; $\triangle APB$ has base \overline{AB} of length $6\sqrt{3}$ and height \overline{PO} of 3, so area of $\triangle PAB$ is $\frac{1}{2} \times 6\sqrt{3} \times 3 = 9\sqrt{3}$. The overlap region has area $2(12\pi - 9\sqrt{3}) = 24\pi - 18\sqrt{3}$, so the shaded area is $36\pi - (24\pi - 18\sqrt{3}) = 12\pi + 18\sqrt{3}$. The area of the rectangle is $18 \times 12 = 216$, so the shaded area occupies $\frac{12\pi + 18\sqrt{3}}{216} = \frac{2\pi + 3\sqrt{3}}{36} = 0.31887\dots$, which rounds to the nearest hundredth as **0.32**.

9. Either Felix or Oscar goes in the red cage; the other goes in the blue cage—this yields 2 options. For each of those two options, there are 4 remaining hamsters. Any of the 4 may be second to occupy the red cage, and after that choice is made any of the 3 remaining ones may be third to occupy the red cage. Now, the last 2 go in the blue cage. Thus, it appears that there are $2 \times 4 \times 3 = 24$ ways of selecting which hamsters will occupy the red cage. However, we could have swapped which hamster was chosen second versus which hamster was chosen third, and we would end up with the same occupancy in each cage. Therefore, we need to divide by 2 to avoid double counting, so we end up with $24/2 = \mathbf{12}$ ways. We could also have done it as 2 (for Felix and Oscar) times the number of ways (combinations) to distribute 4 things (the remaining 4 hamsters) taken 2 at a time (2 to go in the red box) to get $2 \times \frac{4!}{2!2!} = 2 \times \frac{24}{2 \times 2} = \mathbf{12}$ ways.

10. The result does not depend on the shape nor on the size of the triangle. To make the intuition a little easier, let's use a right triangle:



The area of $\triangle APQ$ is $\frac{1}{2} \cdot AP \cdot RQ = \frac{1}{2} \cdot \frac{AP}{AB} \cdot AB \cdot \frac{QR}{BC} \cdot BC = \frac{AP}{AB} \cdot \frac{QR}{BC} \cdot \left(\frac{1}{2} \cdot AB \cdot BC\right)$.

The right end of the equation contains $\frac{AP}{AB} \cdot \frac{QR}{BC}$ times the area of $\triangle ABC$, and expression that

we are given to be $\frac{1}{2}$ the area of $\triangle ABC$. Therefore, $\frac{AP}{AB} \cdot \frac{QR}{BC} = \frac{1}{2}$. We are given that $\frac{PB}{AP} = a$, so $\frac{AP}{AB} = \frac{1}{a+1}$ and

$\frac{PB}{AB} = \frac{a}{a+1}$, with only the former being important to us. Let $q = \frac{QC}{QA}$, which we wish to be at most $\frac{1}{4}$. By

similarity, $q = \frac{QC}{QA} = \frac{BC-QR}{QR} = \frac{BC}{QR} - 1$, so $\frac{QR}{BC} = \frac{1}{q+1}$. Therefore, $\frac{1}{a+1} \cdot \frac{1}{q+1} = \frac{1}{2}$ and $a + 1 = \frac{2}{q+1}$, so

$a = \frac{2}{q+1} - 1 = \frac{1-q}{1+q}$; symmetrically, $q = \frac{1-a}{1+a}$. Because a is restricted to $0 < a < 1$, that means that q is

restricted to $0 < q < 1$. However, we want to know the fraction of values for which $q \leq \frac{1}{4}$. Now, we must be

extremely careful here. It might now appear that the probability is $\frac{1}{4}$ because the interval $0 < q \leq \frac{1}{4}$ is $\frac{1}{4}$ as wide as the interval $0 < q < 1$. However, we do not know that the various values of q are equally likely over these intervals. We are given only that a is randomly chosen—meaning all values in the specified interval are equally likely; q is not called out as random. The region of interest for q is $0 < q \leq \frac{1}{4}$, which

corresponds to $\frac{3}{5} \leq a < 1$, which is $\frac{1-\frac{3}{5}}{1-0} = \frac{2}{5}$ of the allowed interval for the randomly chosen a , so the

probability is $\frac{2}{5}$. [Because this value is not $\frac{1}{4}$, it demonstrates that the various possible values for q are not equally likely—unlike with a . This is to be expected since the relationship between a and q is non-linear.]