MATHCOUNTS Minis

# May 2012 Activity Solutions

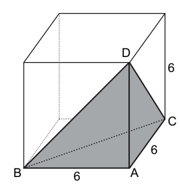
**Warm-Up!**

1. To determine the area of a triangle, we need the length of its base, $b$, and its height, $h$. We are told that each side has length 20 units, so $b = 20$. If we draw the altitude from the vertex angle to the base, we create two 30-60-90 right triangles. The length of the shorter leg is 10 units, and using the properties of 30-60-90 right triangles, we can conclude that the length of the longer leg is $10\sqrt{3}$ units. So, $h = 10\sqrt{3}$. Therefore, the area of the triangle is $\frac{1}{2} \times b \times h = \frac{1}{2} \times 20 \times 10\sqrt{3} = 10 \times 10\sqrt{3} =$ **$100\sqrt{3}$** units$^2$ ≈ **173.21** units$^2$. You may also recall that $A = (s^2\sqrt{3}) \div 4$ for equilateral triangles. Using this formula we get $(20^2\sqrt{3}) \div 4 \approx$ **173.21** units$^2$

2. The area of the shaded region is the difference of the areas of the square and the circle. Since the square has sides of length 10 units, it follows that its area is $s^2 = 10^2 = 100$ units$^2$. The diameter of the circle is also 10 units, which means the radius is 5 units. The area of the circle then is $\pi r^2 = \pi(5^2) = 25\pi$ units$^2$. Therefore, the area of the shaded region is **$100 - 25\pi$** unit$^2$ ≈ **21.46** units$^2$.

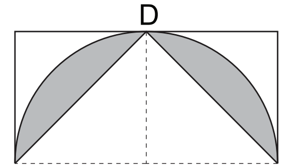
3. The area of the shaded region is the difference of the areas of the circle and the square. Since the square has sides of length 10 units, it follows that its area is $s^2 = 10^2 = 100$ units$^2$. The diagonal of the square is a diameter of the circle. The diagonal of the square is also the hypotenuse of a 45-45-90 right triangle. Based on the properties of 45-45-90 right triangles, we know the length of the diagonal, and therefore the circle's diameter is $10\sqrt{2}$ units. This means the radius of the circle is $5\sqrt{2}$ units, and its area is $\pi r^2 = \pi(5\sqrt{2})^2 = 50\pi$ units$^2$. We see now that the area of the shaded region is **$50\pi - 100$** units$^2$ ≈ **57.08** units$^2$.

4. Tetrahedron ABCD is shown here. The volume of the tetrahedron (which is a pyramid with a triangular base) is $\frac{1}{3} \times B \times h$, where $B$ is the area of the base, and $h$ is the height of the tetrahedron. The area of the base ($\Delta ABC$) is $\frac{1}{2} \times 6 \times 6 = 18$ cm$^2$. Since the height is 6 cm, the volume of the tetrahedron is $\frac{1}{3} \times 18 \times 6 =$ **36** cm$^3$.




**The Problem** is solved during the MATHCOUNTS Mini.

**Follow-up Problems**




5. If we slice the square along diagonal BD and rotate the half containing $\Delta ABD$ clockwise 90° about the original point D, the result is the figure shown here. We can see now that the area of the shaded region is just the area of the semicircle with radius 8 units less the area of the triangle with base and height measuring 16 units and 8 units, respectively. The area of the semicircle is is $\frac{1}{2} \times \pi \times r^2 = \frac{1}{2} \times \pi \times 8^2 = 32\pi$ units$^2$. The area of the triangle is $\frac{1}{2} \times b \times h = \frac{1}{2} \times 16 \times 8 = 64$ units$^2$. That means the area of the shaded region is **$32\pi - 64$** units$^2$ ≈ **36.53** units$^2$.

6. The area of the shaded region is the area of the equilateral triangle less the area of the sector of the circle with central angle measuring 60°. Recall that using the properties of 30-60-90 right triangles we can conclude that the height of the equilateral triangle is $3\sqrt{3}$ units. Therefore, the area of the triangle is $\frac{1}{2} \times b \times h = \frac{1}{2} \times 6 \times 3\sqrt{3} = 9\sqrt{3}$ units$^2$. The area of the sector is 60/360 = 1/6 the area of the circle with a radius of $3\sqrt{3}$ units. The area of the sector then is $1/6 \times \pi \times r^2 = 1/6 \times \pi \times (3\sqrt{3})^2 = (27/6)\pi = (9/2)\pi$ units$^2$. Therefore, the area of the shaded region is **$9\sqrt{3} - (9/2)\pi$** units$^2$ ≈ **1.45** units$^2$

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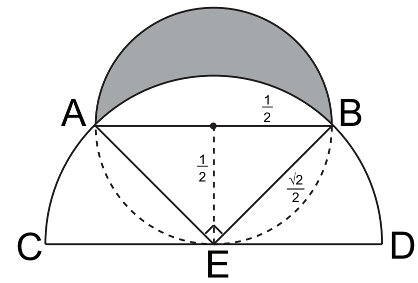


7. The area of the lune is the area of the smaller semicircle less the area of the segment of the larger semicircle intercepting arc AB and bounded by chord AB. Since the AB = 1, the area of the smaller semicircle is $\frac{1}{2} \times \pi \times r^2 = \frac{1}{2} \times \pi \times (\frac{1}{2})^2 = \pi/8$ units$^2$. The area of the segment of the larger semicircle bounded by chord AB is the area of sector of the larger semicircle intercepting arc AB less the area of $\Delta$ABE, as shown, where E is the midpoint of segment CD. Notice that BE = $\sqrt{2}/2$ because it is a radius of the larger semicircle whose diameter we are told is $\sqrt{2}$ units. The segment drawn from point E perpendicular to segment AB intersects the center of the smaller semicircle. Using the Pythagorean Theorem we see that the length of this segment is 1/2 units, which means point E also lies on the smaller circle. Since $\angle$AEB intercepts the diameter of the smaller semicircle, $m\angle$AEB = 90°. That means the area of the sector intercepting arc AB is 90/180 = 1/2 the area of the semicircle. The area of the sector is $\frac{1}{2} \times (\frac{1}{2} \times \pi \times r^2) = \frac{1}{2} \times (\frac{1}{2} \times \pi \times (\sqrt{2}/2)^2) = \frac{1}{2} \times (\frac{1}{4}\pi) = \pi/8$. The area of $\Delta$AEB is $\frac{1}{2} \times b \times h = \frac{1}{2} \times 1 \times \frac{1}{2} = 1/4$ units$^2$. It follows that the area of the segment is $\pi/8 - 1/4$. Therefore, the area of the lune is $\pi/8 - (\pi/8 - 1/4) = \pi/8 - \pi/8 + 1/4 =$ **1/4** units$^2$.

8. The polyhedron is a cube from which two corners have been removed. The length, width and height of the prism are each 4 units. We can determine the volume of the polyhedron using the method demonstrated in the video by finding the difference of the volume of the cube and the combined volume of the two corners that have been removed. The volume of the cube is $l \times w \times h = 4 \times 4 \times 4 = 64$ units$^3$. The volume of each tetrahedral corner that has been removed is $\frac{1}{3} \times B \times h = \frac{1}{3} \times (\frac{1}{2} \times 4 \times 4) \times 4 = \frac{1}{3} \times 8 \times 4 = 32/3$ units$^3$. Therefore, the area of the polyhedron is $64 - (2 \times 32/3) = 64 - (64/3) =$ **128/3** units$^3$ ≈ **42.67** units$^3$.

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