MATHCOUNTS Minis — March 2012 Activity Solutions

Warm-Up!

1. Because these are each arithmetic sequences, we know the difference between consecutive terms for each sequence remains constant. (In other words, the same amount is added to each term to get the next term.)

(a) 11 – 5 = 6, so the common difference is 6. The terms are 5, 11, *11 + 6 or* **17**, *17 + 6 or* **23**, *23 + 6 or* **29**.

(b) Again, the common difference is 6, but we must work backwards. __, __, *5 – 6 or* **–1**, 5, 11. Then __, *–1 – 6 or* **–7**, **–1**, 5, 11. And finally *–7 – 6 or* **–13**, **–7**, **–1**, 5, 11.

(c) Going from 5 to 11, we must add the common difference to 5 a total of four times. The total difference is 6. Dividing this into four equal parts, we see the common difference of the arithmetic sequence is 6/4 = 3/2 = 1.5. Therefore, the sequence is 5, **6.5**, **8**, **9.5**, 11.

2. Because these are each geometric sequences, we know the ratio of consecutive terms for each sequence remains constant. (In other words, the same amount is multiplied by each term to get the next term.)

(a) 18 ÷ 2 = 9, so the common ratio is 9. The terms are 2, 18, *18 × 9 or* **162**, *162 × 9 or* **1458**, *1458 × 9 or* **13,122**.

(b) Again the common ratio is 9, but we must work backwards. __, __, *2 ÷ 9 or* **2/9**, 2, 18. Then __, *2/9 ÷ 9 or* **2/81**, **2/9**, 2, 18. And finally, *2/81 ÷ 9 or* **2/729**, **2/81**, **2/9**, 2, 18.

3. Knowing the first two terms of the arithmetic sequence are 5, 2 tells us the common difference is 2 – 5 = –3 (or we subtract 3 to get from one term to the next). Therefore, the third and fourth terms of the sequence are 2 – 3 = –1 and –1 – 3 = –4, respectively. Now we know –1, –4 are the first two terms of a geometric sequence. The common ratio is –4 ÷ –1 = 4 (or we multiply by 4 to get from one term to the next). Therefore the third term of the geometric sequence is –4 × 4 = –16 and the fourth term is –16 × 4 = **–64**.

4. Knowing the first two terms of the arithmetic sequence are 4, 10 tells us the common difference is 10 – 4 = 6. To get from 4 to 1000, we will need to add 1000 – 4 = 996. This means we will have to start with 4 and add 6 a total of 996 ÷ 6 = 166 times. This will get us to exactly 1000. We will first exceed 1000 by starting with 4 and adding 6 a total of 167 times. The result is 4 + 6(167) = **1006**.

The Problem is solved during the MATHCOUNTS Mini.

Follow-up Problems

5. We can ignore the unknown first term and consider the sequence beginning with −7 and with 106 as the fourth term. We know that to go from −7 to 106 the common difference, d, is added three times. So we have the equation $-7 + 3d = 106$. Solving this equation we see that $3d = 113$, and $d = 113/3$. The fourteenth term of the original sequence is now the thirteenth term when we disregard the first term and consider the sequence beginning with −7. So the thirteenth term of the sequence is $-7 + 12d = -7 + 12 \times (113/3) = -7 + 4 \times 113 = -7 + 452 =$ **445**.

6. Consider that to get from the third term to the sixth term, you would need to multiply by the common ratio 3 times. In other words, the third term is –2, and we are looking for the sixth term, which is $(-2)(r^3)$. We also know $(-2)(r^{12}) = -162$. Since the exponents of r are both multiples of 3, and we'd rather work with smaller numbers, what if we let $y = r^3$? Then $(-2)(r^{12}) = -162$ becomes $(-2)(y^4) = -162$. Dividing both sides of the equation by –2 gets us $y^4 = 81$. And then taking the fourth root, which some people may recognize and is easier than taking a 12th root, we see $y = \pm 3$. Remember we are looking for $(-2)(r^3)$ and $y = r^3$, so we want $(-2)(\pm 3)$. Since all of our terms are negative, we'll use $r^3 = 3$, and the sixth term is **–6**.

7. This situation represents a geometric sequence with a common ratio of 2. (The number of amoebas for the first few days is 1, 2, 4, 8, 16, ...) Using the formula from the Mini (nth term = ar^{n-1}), on the 23rd day, there would be $(1)(2^{22}) = 2^{22}$ amoebas. If 2^{22} amoebas are required to fill the puddle, then $2^{22} \div 2 = 2^{21}$ amoebas are required for the puddle to be half-full. Notice $n - 1 = 21$, so $n = 22$, and we see it would happen on the **22nd day**. This makes sense since the number of amoebas doubled every day... the puddle would be half-full on the day before it was full!

8. To determine the number of distinct arithmetic sequences having 1 as the first term with 91 in the sequence, we first need to determine how far it is from 1 to 91. Since 91 − 1 = 90 and the sequence it to contain only integers, we need to determine the number of ways we can divide 90 into integral parts. In other words, what are the factors of 90? The factors of 90 are 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45 and 90. These are the twelve possible values for the common difference, d. That means there are **12** distinct arithmetic sequences that meet the given conditions.

9. We are told that $y/w = 3$, so $y = 3w$. We have the sequence v, w, x, $3w$, z. Since this is an arithmetic sequence, it follows that x must be $2w$ (since it's the average of its two neighboring terms), the common difference is w, and z then is $4w$. That means that $z/x = (4w)/(2w) =$ **2**.