MATHCOUNTS Minis November 2011 Activity Solutions

Warm-Up!

1. There is a segment of length 1 unit between each pair of adjacent dots in each of the 6 columns of the grid. That's 3 segments per column for a total of $3 \times 6 = 18$ vertical segments. There is a segment of length 1 unit between each pair of adjacent dots in each of the 4 rows of the grid. That's 5 segments per row for a total of $5 \times 4 = 20$ horizontal segments. Therefore, the total number of segments with length 1 unit is $18 + 20 =$ **38** segments.

2. Let's make an organized list to keep track of the parallelograms as we find them. We will categorize each parallelogram based on the number of rows and columns it spans. There are 4 parallelograms that span one row and one column. There are 2 parallelograms that span one row and two columns. There are also 2 parallelograms that span two rows and one column. Finally, there is 1 parallelogram that spans two rows and two columns. That brings the total number of parallelograms in the figure to $4 + 2 + 2 + 1 =$ **9** parallelograms.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems

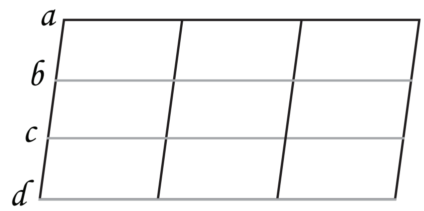
3. Since a word in Quadland can have at most 4 letters, we need to consider words containing 1 letter, 2 letters, 3 letters and 4 letters. We are told that the Quadland language contains 4 different letters, so there are 4 ways to make a 1-letter word. For a 2-letter word, there are 4 choices for the first letter and 4 choices for the second letter. Thus, there are $4 \times 4 = 16$ different 2-letter words. Similarly, to make a 3-letter word, there are 4 possibilities for the first letter, 4 for the second letter and 4 for the third letter. That means it is possible to make $4 \times 4 \times 4 = 64$ different 3-letter words. Finally, to make a 4-letter word there are 4 choices for the first letter, 4 for the second letter, 4 for the third letter and 4 choices for the fourth letter. It follows that there are $4 \times 4 \times 4 \times 4 = 256$ possible 4-letter words. Therefore, the number of different words that are possible in Quadland is $4 + 16 + 64 + 256 =$ **340** words.

4. It seems that organizing and listing possibilities has worked for each problem thus far, so let's try it again. First, notice that the sum $m^2 + n$ must be less than 31. That means the largest possible value for m is 5 since 6^2 is 36, which is greater than 31. Now let's rewrite the inequality in the form $n < 31 - m^2$ and examine the possible values of n when m is 1, 2, 3, 4 or 5.

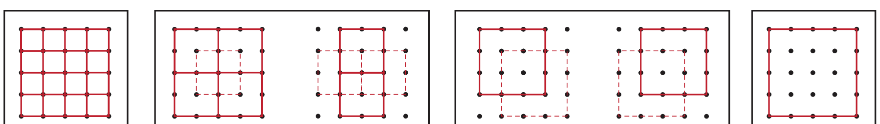
- If $m = 1$, we have $n < 31 - 1^2 \rightarrow n < 30$. The integer n can be any integer from 1 to 29, inclusive.
- Next, if $m = 2$, we have $n < 31 - 2^2 \rightarrow n < 27$. In this case, n can be any integer from 1 to 26, inclusive.
- If $m = 3$, we have $n < 31 - 3^2 \rightarrow n < 22$. Here n can take the value of any integer from 1 to 21, inclusive.
- If $m = 4$, we have $n < 31 - 4^2 \rightarrow n < 15$. In this case, n can be any integer from 1 to 14, inclusive.
- Finally, if $m = 5$, we have $n < 31 - 5^2 \rightarrow n < 6$. The integer n can be any integer from 1 to 5, inclusive.

Therefore, the number of pairs of integers that satisfy the inequality is $29 + 26 + 21 + 14 + 5 =$ **95** pairs.

5. Consider constructing a parallelogram by choosing one of the four horizontal lines (labeled a, b, c and d in the figure) as the base of the parallelogram and one other horizontal line as the top of the parallelogram. Only lines b, c and d can serve as the base of a parallelogram. If line b is chosen to be the base of the parallelogram, there is only 1 option for the top of the parallelogram, which is line a. If line c is selected as the base of the parallelogram, there are 2 options for the top of the parallelogram, which are line a and line b. If line d is the base of the parallelogram, there are 3 options for the top of the parallelogram, which are line a, line b and line c. So it appears that depending on which horizontal line is chosen first, there are 1, 2 or 3 ways to select the second horizontal line. So a pair of horizontal lines can be selected in 1 + 2 + 3 = 6 ways. Similarly, depending on which diagonal line is chosen first, there are 1, 2 or 3 ways to select the second diagonal line. So a pair of diagonal lines can be selected in 1 + 2 + 3 = 6 ways. Therefore, we have that the number of ways to construct a parallelogram by selecting two horizontal lines and two diagonal lines in the figure is (1 + 2 + 3)(1 + 2 + 3) = (6)(6) = 36 ways.



6. This problem is a bit trickier than the parallelogram problem from the video. First, we can't arbitrarily select two horizontal lines and two vertical lines to form a square. Once you select two horizontal lines the vertical lines have to be selected in such a way that the horizontal length is the same as the vertical length. Another consideration is that squares can be formed in the grid by combining a pair of horizontal lines with a pair of vertical lines and by combining a pair of diagonal lines with a positive slope with a pair of diagonal lines with a negative slope. Let's first determine how many squares can be created using only horizontal and vertical lines. We begin by making an organized list. There are 16 1-by-1 squares. There are nine 2-by-2 squares. There are four 3-by-3 squares. And there is one 4-by-4 square. These squares are shown below.



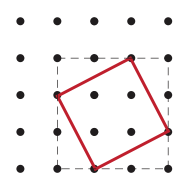
1 × 1
16 squares

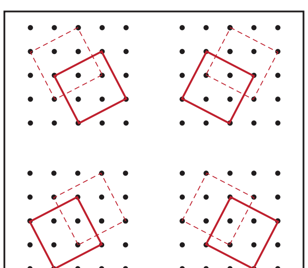
2 × 2
9 squares

3 × 3
4 squares

4 × 4
1 square

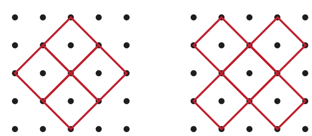
Now we will organize and list the possible diagonal squares. The figure to the right shows a diagonal square (solid line) inside a 3-by-3 region (dotted line) of the grid. To determine the side lengths of the square we can use the Pythagorean Theorem. Notice the right triangles in each corner of the 3-by-3 region. Each of these triangles has a short leg with length 1 unit, a long leg with length 2 units and a hypotenuse that is a side of the diagonal square. We know the length of the hypotenuse is $c = \sqrt{(a^2 + b^2)} \rightarrow c = \sqrt{(1^2 + 2^2)} \rightarrow c = \sqrt{5}$. We can form 7 more squares of this size on our grid, as shown below.



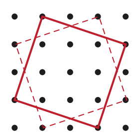


$\sqrt{5} \times \sqrt{5}$
8 squares

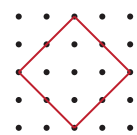
But we need to determine all the possible side lengths for these diagonal squares to ensure that we don't miss any. To find all the possible side lengths, consider that the side of the square will be the hypotenuse of a right triangle and we can write $c = \sqrt{(a^2 + b^2)}$, where *c* is the length of the hypotenuse and *a* and *b* are the lengths of the shorter and longer legs of the triangle, respectively. If $a = 1$, we can form right triangles such that $b = 1$, $b = 2$ or $b = 3$. It doesn't work if $b = 4$ because the square would extend beyond our grid. That gives us squares with side lengths $c = \sqrt{(1^2 + 1^2)} = \sqrt{2}$, $c = \sqrt{(1^2 + 2^2)} = \sqrt{5}$ and $c = \sqrt{(1^2 + 3^2)} = \sqrt{10}$. If $a = 2$, we can form right triangles such that $b = 1$ or $b = 2$. Again, it doesn't work if $b = 3$ because the square would extend beyond our grid. That gives us squares with side lengths $c = \sqrt{(2^2 + 1^2)} = \sqrt{5}$ and $c = \sqrt{(2^2 + 2^2)} = 2\sqrt{2}$. Below are the squares that can be formed with sides of lengths $\sqrt{2}$ units, $\sqrt{10}$ units and $2\sqrt{2}$ units (the 8 squares with sides of length $\sqrt{5}$ units were shown previously).



$\sqrt{2} \times \sqrt{2}$
9 squares



$\sqrt{10} \times \sqrt{10}$
2 squares



$2\sqrt{2} \times 2\sqrt{2}$
1 square

Finally, that brings the total number of squares to 16 + 9 + 4 + 1 + 8 + 9 + 2 + 1 = **50** squares.

Further Exploration

7. We can solve this problem in the same manner as the problem solved by Richard in the video. There are 8 horizontal lines and 10 diagonal lines in the figure. That means there are $(8 \times 7)/2 = 28$ ways to choose two horizontal lines and $(10 \times 9)/2 = 45$ ways to choose a pair of diagonal lines. So, the number of parallelograms in the figure is $28 \times 45 =$ **1260** parallelograms.