MATHCOUNTS Minis May 2011 Activity Solutions

Warm-Up!

1. The area of rectangle ABCD is 27(11) = 297 sq units. If we subtract the area of the triangular region that is removed from the area of rectangle ABCD, the result is the area of pentagon ABEFD. Now CF = CD – FD = 27 – 15 = 12 units, and EC = BC – BE = 11 – 6 = 5 units. Thus the area of ΔCEF is ½(12)(5) = 30 sq units. That means the area of pentagon ABEFD is 297 – 30 = **267** sq units.

2. We are told that segment XY intersects the center of the square, which means CY = AX = 2 units. We can now determine that DY = DC – YC = 10 – 2 = **8** units. Now quadrilateral AXYD is a trapezoid with bases $\overline{DY}$ and $\overline{AX}$. The area of a trapezoid is just the average of the lengths of its bases times its height. Thus, the area of trapezoid AXYD is ½(2 + 8)(10) = ½(10)(10) = 5(10) = **50** sq units.

3. We know that the $m\angle$APB = $m\angle$DPC because opposite angles are congruent. Also, we know that $m\angle$PCD = $m\angle$PBA since alternate interior angles are congruent. From this we conclude that ΔCDP ~ ΔBAP and $\frac{CP}{14-CP} = \frac{8}{20}$. After cross-multiplying, the resulting equation is CP(20) = (14 − CP)(8) → CP(20) = 112 − CP(8) → CP(28) = 112 → CP = **4** units.

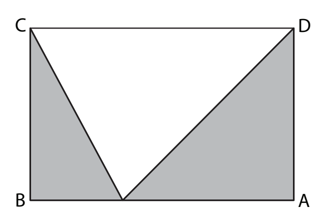
4. The area of a triangle is determined by taking half the product of the length of its base and its height. The two triangles formed by the median have the same height and the lengths of their bases are equal, therefore the area of the two triangles is equal.

5. In order to determine the area of ΔPQR we need to know the length of the base and the height. We know PQ = 3, and segment PQ is a base of ΔPQR. The height of ΔPQR is the same as the height of the trapezoid. We are told that the area of trapezoid PQRS is 24 sq units and that the lengths of the bases are 3 units and 9 units. We can use this information to determine the height, h, of the trapezoid. We have ½(3 + 9)h = 24 → ½(12)h = 24 → 6h = 24 → h = 4 units. Now we see that the area of ΔPQR is ½(3)(4) = ½(12) = **6** sq units.

The Problem is solved in the MATHCOUNTS Mini.

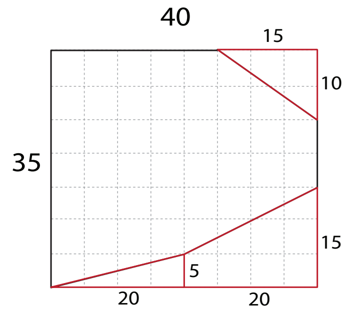
Follow-up Problems

6. The figure appears to be a rectangle from which side $\overline{CD}$ has been removed and two of the three interior triangles have been shaded, as shown. If we determine the area of rectangle ABCD and subtract from it the area of the unshaded triangle, the result will be the area of the shaded region. The area of the rectangle is (8)(12) = 96 cm^2. The area of the unshaded triangle is ½(12)(8) = (6)(8) = 48 cm^2. That means the area of the shaded region is 96 − 48 = **48** cm^2.

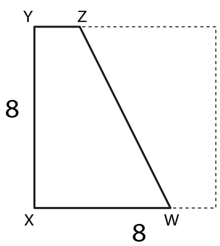


You may also have recognized that if we rotate the two shaded triangles 180° and translate them so that segment BC and segment AD perfectly overlap, the resulting figure fits the unshaded region exactly. The combined area of the two shaded triangles is the same as the area of the unshaded triangle with base = 12 cm and height = 8 cm.

7. To find the area of this irregular-shaped region, let's first determine the area of the large rectangular region. The area is 40(35) = 1400 sq ft. Next, we can break up the area outside the irregular shape, but inside the rectangular region, into several shapes with easily determined areas, as shown. The triangle in the upper right corner of the rectangular region has an area of ½(10)(15) = 75 sq ft. The triangle in the lower left corner has an area of ½(20)(5) = 50 sq ft. The area of the trapezoid in the lower right corner is ½(5 + 15)(20) = ½(20)(20) = 200 sq ft. That means the area of the irregular-shaped region is 1400 − (75 + 50 + 200) = 1400 − 325 = **1075** sq ft.

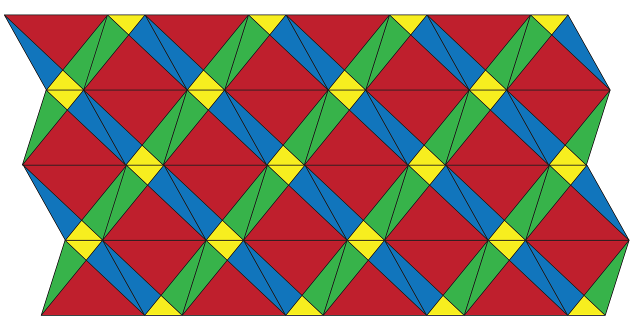


8. The area of △ABC is the sum of the areas of △CAP and △CBP. We are told that the area of △CAP is 40 sq units. We do not know the length of segment AP, so let's call it $2x$. That means that $½(2x)h = 40$. In other words, $xh = 40$. We are told that AP:PB = 1:2. If AP = $2x$ then PB = $4x$ and the area of △CBP is $½(4x)h = 2xh$. We know that $xh = 40$ sq units so, $2xh = 80$ sq units. Now we can see that the area of the △ABC is equal to 40 + 80 = **120** sq units.



9. No matter what the lengths are for $\overline{WX}$ and $\overline{YZ}$, their sum will always be 8 units. The area of trapezoid WXYZ is ½(WX + YZ)(8) = ½(8)(8) = 32 sq units. Also see that if we tesselate trapezoid WXYZ to create trapezoid W′X′Y′Z′, then put the two figures together, as shown, they would form a square. So, no matter what the lengths are of the bases of this trapezoid, as long as the sum of the lenghts is 8 units, the area of the trapezoid will **32** sq units, half the area of the square.

10. Harvey's diagram is a tesselation of the plane with copies of the original trapezoid. We are asked for the area of one blue triangle. In the tesselation, the diagonals of the trapezoids line up conveniently to form parallel strips in two directions. In each direction, the strip's widths have ratio 3:2 (from the bases of the trapezoid). Each color corresponds to one possible pairing of intersections of types of strips. For example, the intersection of the wide strips that go up from left to right with the narrow strips that go down from left to right are entirely blue. Those that are intersections of narrow strips are yellow, and so on. So, when we tile the whole plane with these colored trapezoids, the blue trapezoids are 2/5 of 3/5 of the whole plane. Each trapezoid is colored in the same ratios, so the blue of each trapezoid is 2/5 of 3/5 of each trapezoid.



Follow-up Problems

11. The height, h_1, of ΔAPF is equivalent to that of ΔBPF. Using the notation from the video we have $\frac{[\Delta APF]}{[\Delta BPF]} = \frac{\frac{1}{2}AF(h_1)}{\frac{1}{2}BF(h_1)} = \frac{AF}{BF}$.

12. Similarly, the height, h_2, of ΔACF is equivalent to that of ΔBCF, and $\frac{[\Delta ACF]}{[\Delta BCF]} = \frac{\frac{1}{2}AF(h_2)}{\frac{1}{2}BF(h_2)} = \frac{AF}{BF}$.

13. Let x = ½(h_1), and let y = ½(h_2). It follows that $\frac{[\Delta ACP]}{[\Delta BCP]} = \frac{[\Delta ACF]-[\Delta APF]}{[\Delta BCF]-[\Delta BPF]} = \frac{x AF - y AF}{x FB - y FB} = \frac{(x-y)AF}{(x-y)FB} = \frac{AF}{FB}$

14. As in problems 11 through 13, we can show that $\frac{[\Delta BPD]}{[\Delta CPD]} = \frac{BD}{DC}$ and $\frac{[\Delta BAD]}{[\Delta CAD]} = \frac{BD}{DC}$ which means that $\frac{[\Delta ABP]}{[\Delta ACP]} = \frac{BD}{DC}$. Likewise, since $\frac{[\Delta CBE]}{[\Delta ABE]} = \frac{CE}{EA}$ and $\frac{[\Delta CPE]}{[\Delta APE]} = \frac{CE}{EA}$, we have that $\frac{[\Delta BCP]}{[\Delta ABP]} = \frac{CE}{EA}$. It follows that $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{\cancel{[\Delta ACP]}}{\cancel{[\Delta BCP]}} \cdot \frac{\cancel{[\Delta ABP]}}{\cancel{[\Delta ACP]}} \cdot \frac{\cancel{[\Delta BCP]}}{\cancel{[\Delta ABP]}} = 1$.