MATHCOUNTS® Minis

April 2020 Activity Solutions

Warm-Up!

1. We are looking for the units digit when we multiply all the odd numbers from 1 to 2015, inclusive. If we multiply 1 × 3 × 5 × 7 × 9, and look only at the units digit, we see this product has a units digit of 5. Because the same digits are multiplied, the product of the odd numbers from 11 to 19 will have a units digit of 5, as will the product of the odd numbers from 21 to 29 and from 31 to 39, and so on. Therefore we know 1 × 3 × 5 × ⋯ × 2009 will have a units digit of 5, since 5 multiplied by itself, any number of times, will result in a units digit of 5. The only numbers left to multiply, then, are 2011, 2013 and 2015. Multiplying these three numbers will give us the same units digit as multiplying 1 × 3 × 5, which is 5. The units digit for the product 1 × 3 × 5 × ⋯ × 2015, therefore, also will be **5**.

2. Figure 1 has 1 dot. Figure 2 has 3 dots, which is 2 more that the previous figure. Figure 3 has 6 dots, which is 3 more than the previous figure. Finally, Figure 4 has 10 dots, which is 4 more than the previous figure. Notice the pattern shown in this table. Therefore, Figure 10 has 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = **55** dots. The numbers of dots in each figure form a sequence of numbers commonly referred to as the Triangular Numbers. There is a formula to determine the sum of the first n positive integers. It is $1 + 2 + 3 + 4 + \cdots + n = n(n + 1)/2$. So, in this case, the number of dots in Figure 10, which represents the tenth Triangular Number, is 10(11)/2 = 110/2 = **55** dots.

FIGURE	DOTS
1	1
2	1 + 2 = 3
3	1 + 2 + 3 = 6
4	1 + 2 + 3 + 4 = 10
⋮	⋮
n	$1 + 2 + 3 + 4 + \cdots + n$

3. To evaluate (4 – 3) + (5 – 4) + (6 – 5) + (7 – 6) + ⋯ + (2010 – 2009), we need to first evaluate the expression in each set of parentheses. The value of the expression in each of the 2010 – 4 + 1 = 2007 sets of parentheses is 1. So, the sum is 1 × 2007 = **2007**.

4. There are three triples that add up to 6. Let's examine these three cases:
   Case 1: The triple 2, 2 and 2 can be ordered in 1 way.
   Case 2: The triple 1, 1 and 4 can be ordered in 3!/2 = 3 ways.
   Case 3: The triple 1, 2 and 3 can be ordered in 3! = 6 ways.
These three cases yield a total of 1 + 3 + 6 = **10** ordered triples.

The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. Let's, first, try a few different combinations of positive odd integers x, y and z to see what the remainder is when the sum of their squares is divided by 4. The table shows four such scenarios. It appears that for all positive odd integers x, y and z, when the sum of their squares is divided by 4, the remainder will always be 3. Let's see if we can prove this algebraically. For nonnegative integers u, v and w, let $x = 2u + 1$, $y = 2v + 1$ and $z = 2w + 1$, since x, y and z are odd integers. Then the sum of the squares of x, y and z is $x^2 + y^2 + z^2 = (2u + 1)^2 + (2v + 1)^2 + (2w + 1)^2 = 4u^2 + 4u + 1 + 4v^2 + 4v + 1 + 4w^2 + 4w + 1 = 4(u^2 + u + v^2 + v + w^2 + w) + 3$. This proves that the sum of the squares of all positive odd integers x, y and z will always leave a remainder of **3** when divided by 4.

x	y	z	$x^2 + y^2 + z^2$	$\frac{x^2 + y^2 + z^2}{4}$
1	1	1	3	0 R.3
1	1	3	11	2 R.3
1	3	3	19	4 R.3
1	3	5	35	8 R.3

6. When the dots are connected, triangles are created, as shown. Instead of looking at the line segments, let's look at the number of shaded triangles in each figure.

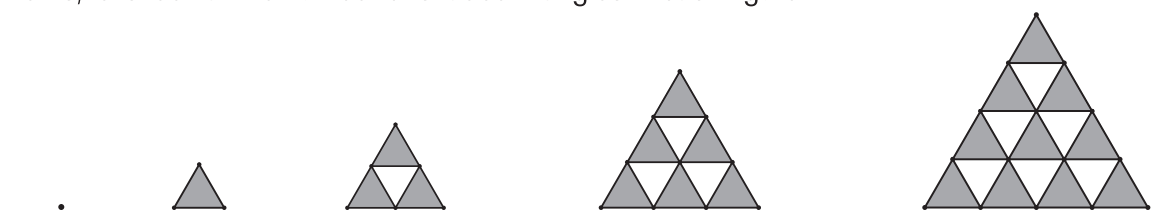

Figure 1 Figure 2 Figure 3 Figure 4 Figure 5

Figure 1 has no triangles. Figures 2, 3, 4 and 5 each have 1, 3, 6 and 10 shaded triangles, respectively. The sequence representing the number of shaded triangles in each figure is 0, 1, 3, 6, 10, ... Recall, that the sequence representing the number of dots in each figure is 1, 3, 6, 10, 15, ... Notice that these two sequences are the same, except the terms are one-off because the first term of the sequence representing the shaded triangles is 0. It follows, then, that in Figure 10 there are $0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ shaded triangles. Each shaded triangle has a perimeter of 3. That means the length of all the segments in Figure 10 is $45 \times 3 =$ **135** units.

7. To determine the number of distinct arithmetic sequences having 1 as the first term with 91 in the sequence, we first need to determine how far it is from 1 to 91. Since $91 - 1 = 90$ and the sequence contains only integers, we need to determine the number of ways we can divide 90 into integral parts. In other words, what are the factors of 90? The factors of 90 are 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45 and 90. These are the twelve possible values for the common difference, *d*. That means there are **12** distinct arithmetic sequences that meet the given conditions.

8. Let's start with the case in which the sum includes five 1. Then we'll reduce the number of 1s while listing the sums for each case.

Case 1: In the case of five 1s, there is only 1 sum: $13 + 1 + 1 + 1 + 1 + 1$.

Case 2: In the case of four 1s, the other two odd numbers must add up to 14. There are three possibilities: $11 + 3$, $9 + 5$ and $7 + 7$. This case yields 3 sums.

Case 3: In the case of three 1s, the other three odd numbers must add up to 15. There are three possibilities: $9 + 3 + 3$, $7 + 5 + 3$ and $5 + 5 + 5$. This case yields 3 sums.

Case 4: In the case of two 1s, the other four odd numbers must add up to 16. There are two possibilities: $7 + 3 + 3 + 3$ and $5 + 5 + 3 + 3$. This case yields 2 sums.

Case 5: In the case of one 1, the other five odd numbers must add up to 17. There is one possibility: $5 + 3 + 3 + 3 + 3$. This case yields 1 sum.

Case 6: In the case of no 1s, there is one possibility with six odd numbers that add up to 18: $3 + 3 + 3 + 3 + 3 + 3$. This case yields 1 sum.

These six cases yield a total of $1 + 3 + 3 + 2 + 1 + 1 =$ **11** sums.