MATHCOUNTS Minis April 2011 Activity Solutions

Warm-Up!

1. The arithmetic sequence of the first 31 positive integers consists of the 31 integers 1, 2, 3, ..., 30, 31. There will be the first 15 terms, then the 16th term, which is **16**, and then the last 15 terms.

2. An arithmetic sequence with 21 terms (which is an odd number of terms), will have an exact middle term. There will be the first 21 ÷ 2 = 10.5 → 10 terms, then the 11th term will be the middle term, and then there will be the last 10 terms. The common difference of this arithmetic sequence is 4. To get from the 1st term, 3, to the 11th term, we will need to add the common difference 10 times: 3 + 10(4) = **43**.

3. The common difference of this arithmetic sequence is 33 – 31 = 2, since 31 and 33 are consecutive terms of the sequence. Because this sequence has an even number of terms, 14, there will be the first 14 ÷ 2 = 7 terms and the second 7 terms. Therefore, 31 is the 7th term and 33 is the 8th term. To get from the 7th term to the 14th term, we need to add the common difference 7 times: 31 + 7(2) = **45**.

4. The mean of the first 50 positive, even integers will be the sum of the integers divided by 50. This sequence is 2, 4, 6, ..., 98, 100. If we look at the integer pairs, moving from the outside members of the list of 50 integers toward the middle, we have 2 and 100; 4 and 98; 6 and 96; ..., 50 and 52. Notice that the sum of each pair is 102 and there are 25 pairs. Their sum is then (102)(25) = 2550. Dividing this by 50 gives us a mean of **51**. (Notice that there is no middle term for this arithmetic sequence, but our answer is equal to the mean of the two middle-most terms.)

5. As we found in the last problem, the middle two numbers are **50** and **52**.

6a. The average score of the team members' Sprint Round scores is (7 + 8 + 21 + 24) ÷ 4 = 60 ÷ 4 = **15**.

6b. The boys' scores were 8 points and 7 points below the team average, for a total of **15** points.

6c. The girls' scores were 6 points and 9 points above the team average, for a total of **15** points.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems

7. If the sum of 7 consecutive odd integers is 273, and the middle term is the average of those integers, the middle term is 273 ÷ 7 = 39. Since these are consecutive odd integers, their common difference is 2. So going from the middle term (or 4th term) to the 7th term, we need to add the common difference 3 times: 39 + 3(2) = **45**.

8. If the sum of the first five terms of an arithmetic sequence is 75, then the middle term, or 3rd term, is 75 ÷ 5 = 15. The extended sequence with 11 terms must have a middle term, or 6th term, of 363 ÷ 11 = 33. The difference between the 6th term and 3rd term is 33 – 15 = 18, which is the common difference added on 3 times. So the common difference is 18 ÷ 3 = 6. Going from the 3rd term to the 1st term would mean taking the common difference off twice:
15 – 2(6) = **3**.

9. The middle term of the first 9 consecutive odd integers will be the 5th odd integer, which is 9. This is also the mean of the first 9 consecutive odd integers, so their sum is 9 × 9 = 81. This is 3 more than the sum of 6 consecutive even integers. Their sum is then 78. Their average value is 78 ÷ 6 = 13. So the 3rd and 4th terms are 12 and 14, and the 6th term is **18**.

10. If we assume there will only be two consecutive integers with this sum, then we know 210 ÷ 2 = 105 is the value *between* them. However, it's impossible for 105 to be *between* two consecutive integers. What if we assume we can find three consecutive positive integers with a sum of 210? Then the middle term - and with an odd number of terms, there is an exact middle term - will be 210 ÷ 3 = 70, and the terms are 69, 70 and 71, with 71 being the largest. This shows us that with an odd number of terms, that odd number must be a factor of 210 in order to get an exact middle term that is an integer. The odd factors of 210 are 1, 3, 5, 7, 15, 21, 35 and 105. If there were 105 terms, then the middle (53rd) term would be 210 ÷ 105 = 2, and we can see that spreading out 52 integers in both directions will result in terms that are negative integers. What if there were 35 terms? The middle (18th) term would be 210 ÷ 35 = 6, and again we would get terms that are negative integers when spreading out 17 integers in both directions. Similarly, assuming there are 21 terms will result in a middle (11th) term of 10 and a smallest term of 0. However, with 15 terms, the middle (8th) term is 210 ÷ 15 = 14, the smallest term is 7 and the largest term is 21. This takes care of the possibilities with an odd number of terms. However, just because 2 terms didn't work, doesn't mean there couldn't be an even number of terms...

Perhaps there can be four terms... 210 ÷ 4 = 52.5. Being that four terms is an even number of terms, this 52.5 must be *exactly between* two consecutive integers, and it is: 52 and 53. The set would be 51, 52, 53 and 54, with 54 being the largest. As we can see, this even number cannot be a factor of 21, or it will result in a mean which is an integer and can't be *between* two consecutive integers. In fact, the result of dividing 210 by this even number must result in a number that is halfway between two consecutive integers. Is there an even number larger than 15 that works (since we know 15 is the largest we've come up with so far)? After seeing that 16 and 18 don't work, we see 20 does. Notice 210 ÷ 20 = 10.5, meaning that the 10th term is 10, the 11th term is 11, the first term is 1 (positive) and the last term is 20. If we try any larger number of even terms, we would run into the problem of getting smaller middle values and extending into the negative integers. The answer then is **20** which results from 20 consecutive integers starting with 1.

11. The sum of the interior angles of an octagon is 1080 degrees. Being that there are 8 angles, the average/middle value must be 1080 ÷ 8 = 135. Given that the problem says this is an increasing sequence, we know we can't have an equiangular octagon with 8 angles each measuring 135 degrees. But we have: __ __ __ __ __ __ __ __ .
(135)

As we saw in the video, if our 4th term is smaller than the average by 1, the common difference is 2 and we need to ensure the 4th term minus three of these differences is positive. Additionally, we need to ensure the 5th term plus three of these differences is less than 180 (to keep the figure convex). This upper limit is what we need to worry about (since 135 is closer to 180 than 0). We see that having 4th and 5th terms of 134 and 136; 133 and 137; and so on will work up to a certain point. Where is that point? If we add x to 135 to get the 5th angle, then the common difference of the sequence is $2x$. Then the 6th angle is $135 + 3x$, the 7th angle is $135 + 5x$ and the 8th angle is $135 + 7x$ (which we see is also the 5th angle plus 3 of these differences of $2x$). We need $135 + 7x < 180$. Subtracting 135 from both sides and then dividing by 7 yields $x < 6.43..$, so $x = 6$ is the largest value that works. Remember, this was the amount to add to 135 to get the 5th angle. Therefore, we could add 1, 2, 3, 4, 5 or 6, and there must be **6** possible sequences.

12. Assume each of your first 6 scores were 183. We would need enough “extra points” to raise each of those games by 7 points (which is a total of 42 points) as well get the 190 points necessary for the 7th game to maintain the average of 190. That means the 7th game’s score would need to be 190 + 42 = **232** points.

13. For the original set of 10 numbers, let’s let the average value be x. This tells us the sum of the numbers is $10x$. Removing the largest number results in an average of $x - 1$ and a sum of $9(x - 1) = 9x - 9$ for the 9 remaining numbers. The largest number is then $10x - (9x - 9) = x + 9$. Removing the smallest number results in an average of $x + 2$ and a sum of $9(x + 2) = 9x + 18$. The smallest number is then $10x - (9x + 18) = x - 18$. The positive difference between these two numbers is $(x + 9) - (x - 18) = x + 9 - x + 18 =$ **27**.